QUASI-GEODESICS

Definition 6.12. A path $\alpha : I \to X$ in a geodesic metric space X is a (λ, μ)-*quasi-geodesic*, where $\lambda \geq 1, \mu \geq 0$, if for all $t, s \in I$,

$$length(\alpha([t,s])) \leq \lambda d(\alpha(t), \alpha(s)) + \mu$$

Proposition 6.2. *Let X be a δ-hyperbolic metric space. There exist constants $L = L(\lambda, \mu), M = M(\lambda, \mu)$ such that if $x, y \in X$, $\alpha : I \to X$ is a (λ, μ)-quasi-geodesic with endpoints x, y and $\gamma = [x, y]$ then*

$$\gamma \subset N_L(\alpha), \ \alpha \subset N_M(\gamma)$$

Proof. We show first the existence of L. Let $a \in \gamma$ such that $d(a, \alpha) = D$ is maximum. Let $a_1 \neq a_2 \in \gamma$ with

$$d(a, a_1) = d(a, a_2) = D$$

and let $\alpha(t), \alpha(s)$ points in α realizing $d(a_1, \alpha), d(a_2, \alpha)$, respectively. We consider the path

$$\beta = [a_1, \alpha(t)] \cup \alpha([t,s]) \cup [a_2, \alpha(s)]$$

Clearly $d(a,\beta) \geq D/2$.

We pick points $x_1 = \alpha(t), x_2, ..., x_{n-1} = \alpha(s)$ such that $d(x_i, x_{i+1}) = 1$ for $i = 1, ..., n-3$ and $d(x_{n-2}, x_{n-1}) \leq 1$. By lemma 6.1

$$d(a, [a_1, \alpha(t)] \cup [x_1, x_2] \cup ... \cup [x_{n-2}, x_{n-1}] \cup [a_2, \alpha(s)]) \leq (log_2(n) + 1)\delta$$

and

$$(log_2(n) + 1)\delta \geq \frac{D}{2} - 1 \Rightarrow (2n)^\delta \geq 2^{\frac{D}{2}-1}$$

Since $n - 2 \leq length\,(\alpha([t,s]))$ and $length\,(\alpha([t,s])) \leq 4D\lambda + \mu$ we obtain:

$$(8D\lambda + 2\mu + 4)^\delta \geq 2^{\frac{D}{2}-1}$$

which gives a bound L for D that depends only on λ, μ (and δ).

We show now the existence of M. Let $x = \alpha(s)$. By a continuity argument there is some $y \in \gamma$ such that y is at distance less than L from $\alpha(s_1)$ and $\alpha(s_2)$ with $s_1 \leq s \leq s_2$. It follows that

$$length(\alpha([s_1, s_2]) \leq 2L\lambda + \mu,$$

therefore

$$d(x, \gamma) \leq 2L(\lambda + 1) + \mu$$

so we may take $M = 2L(\lambda + 1) + \mu$.

□

Corollary 6.3. *Let X be a δ-hyperbolic metric space and let Y be a geodesic metric space quasi-isometric to X. Then Y is hyperbolic.*

Proof. Let Δ be a geodesic triangle in Y. If $f : Y \to X$ is a quasi-isometry $f(\Delta)$ is contained in a finite neighborhood of a (λ, μ) quasi-geodesic triangle Δ' in X, where λ, μ depend only on f. By proposition 6.2 Δ' is ϵ-thin for some $\epsilon = \epsilon(\lambda, \mu, \delta) \geq 0$. But then Δ is also δ'-thin for some δ' that depends only on δ and f. □

Hyperbolic Groups

Definition 6.13. Let $G = \langle S \rangle$ where S is finite. We say that G is *hyperbolic* if the Cayley graph $\Gamma = \Gamma(S, G)$ is a hyperbolic metric space.

Remark 6.1. By corollary 6.3 if $G = \langle S_1 \rangle = \langle S_2 \rangle$ with S_1, S_2 finite then $\Gamma(S_1, G)$ is hyperbolic if and only if $\Gamma(S_2, G)$ is hyperbolic, so the definition above does not depend on the generating set S.

We note that if a group G is not finitely generated then for $S = G$, $\Gamma(S, G)$ is bounded, hence hyperbolic. So one can not extend in any reasonable way the definition of hyperbolicity to groups that are not finitely generated.

Examples. 1. Finitely generated free (or virtually free) groups are hyperbolic.

2. Groups acting discretely and co-compactly on $\mathbb{H}^n$ are hyperbolic.

3. $\mathbb{Z}^2$ is *not* hyperbolic.

4. A finite presentation $\langle S|R\rangle$ is said to satisfy condition $C'(\frac{1}{7})$ if for any two cyclic permutations r_1, r_2 of words in $R \cup R^{-1}$ any common initial subword w of r_1, r_2 has length $|w| \leq \frac{1}{7}\min\{|r_1|, |r_2|\}$. It can be shown that $C'(\frac{1}{7})$-groups are hyperbolic. As an example the group

$$G = \langle a, b, c, d | abcdbadc\rangle$$

satisfies the $C'(\frac{1}{7})$ condition, so it is hyperbolic.

5. A theorem of Gromov-Olshanskii shows that 'statistically most groups are hyperbolic': Given $p, q \in \mathbb{N}$ consider all presentations of the form

$$\langle a_1, ..., a_p | r_1, ..., r_q\rangle$$

where the r_i's are cyclically reduced words of the a_j's. Let's denote by $N(t, \lambda t)$ (where $\lambda > 1$) all presentations of this type such that for all i,

$$t \leq |r_i| \leq \lambda t$$

We denote N_h the presentations of hyperbolic groups among those. Then

$$\lim_{t\to\infty} \frac{N_h}{N(t, \lambda t)} = 1$$

Definition 6.14. A *Dehn presentation* of a group G is a finite presentation $\langle S|R\rangle$ such that every reduced word $w \in F(S)$ which is equal to the identity in G contains more than half of a word in R.

Remark 6.2. If $\langle S|R\rangle$ is a Dehn presentation then the word problem for $\langle S|R\rangle$ is solvable. Indeed if w is a word we check if it contains more than half of a relation in R. If not then $w \neq 1$. Otherwise $w = w_1uw_2$ for some $uv \in R$ with $|v| < |u|$. Then $w = w_1v^{-1}w_2$ so we replace w by $w_1v^{-1}w_2$ and we repeat. Since the length decreases this procedure terminates in finitely many steps.

Theorem 6.5. *Let $G = \langle S\rangle$ be a hyperbolic group. Then G has a Dehn presentation. In particular G is finitely presented and the word problem for G is solvable.*

Proof. Assume that triangles in $\Gamma = \Gamma(S, G)$ are δ-thin for $\delta \in \mathbb{N}$. We set

$$R = \{w \in F(S) : |w| \leq 10\delta,\ w \underset{G}{=} 1\}$$

We claim that $\langle S|R\rangle$ is a Dehn presentation for G. We will show that if $w \in F(S)$ is word such that $w \underset{G}{=} 1$ then w contains more than half of a word in R. We remark that this is trivially true if $|w| \leq 10\delta$. We see w as a closed path of length $n = |w|$ in the Cayley graph Γ, $w : [0, n] \to \Gamma$, $w(0) = w(n) = e$. If w contains a subword u of length $\leq 5\delta$ which is not geodesic then there is v with $|v| < |u|$ such that $uv \in R$, so w contains more than half of a relator and we are done. Otherwise let $t \in \{0, 1, 2, ..., n\}$ be such that $d(w(t), e)$ is maximum. We consider the triangles:

$$[e, w(t), w(t - 5\delta)], [e, w(t), w(t + 5\delta)]$$

Since these two triangles are δ-thin and $d(w(t), e) > 5\delta$ we have that

$$d(w(t - 2\delta), w(t + 2\delta)) \leq 2\delta$$

so the subword of length 4δ, $[w(t - 2\delta), w(t + 2\delta)]$ is not geodesic. It follows that w contains more than half of a word in R.

□

Proposition 6.3. *Let G be a hyperbolic group. Then G has finitely many conjugacy classes of elements of finite order.*

Proof. Let $\langle S|R\rangle$ be a Dehn presentation of G. Let g be an element of finite order and let w be an element of the conjugacy class of g of minimal length. Then $w^n = 1$ so the word w^n contains more than half of a relation $r \in R$. We claim that

$$|w| \leq \frac{|r|}{2} + 2$$

Suppose not. We remark that w is cyclically reduced. We have then that $r = r_1 r_2$, with $|r_1| > |r_2|, |r_1| \leq \frac{|r|}{2} + 2$ and $w = utv$, $r_1 = vu$ for some words r_1, r_2, v, t, u where all the previous expressions are reduced. Then $u^{-1}wu = tvu = tr_1$ is in the conjugacy class of g. We have that $tr_1 = tr_2^{-1}$ and

$$|tr_2^{-1}| \leq |t| + |r_2| < |t| + |r_1| = |w|$$

which is a contradiction since w is an element of the conjugacy class of g of minimal length. We remark now that there are finitely many words w of length less than

$$\max\{\frac{|r|}{2} + 2 : r \in R\}$$

so there are finitely many conjugacy classes of elements of finite order. □

We turn now our attention to the conjugacy problem. We recall that if $g \in G = \langle S \rangle$ we denote by $|g|$ the length of a shortest word on S representing g.

Lemma 6.2. *Let $G = \langle S|R \rangle$ be δ-hyperbolic (so triangles in $\Gamma(S,R)$ are δ-thin). If $g_1 \in G$ is conjugate to g_2 then there is some $x \in G$ such that $g_2 = xg_1x^{-1}$ and*

$$|x| \leq (2|S|)^{2\delta+|g_1|} + |g_1| + |g_2|$$

Proof. Let x be a word of minimal length such that $g_1 = xg_2x^{-1}$. Let's say that $x = x_1...x_n$ with $x_i \in S \cup S^{-1}$. We have then

$$|(x_1...x_i)^{-1}g_1(x_1...x_i)| \leq 2\delta + |g_1|$$

for all i with $|g_1| \leq i \leq n - |g_2|$. If

$$|x| \geq (2|S|)^{2\delta+|g_1|} + |g_1| + |g_2| + 1$$

then there are $i < j$ such that

$$(x_1...x_i)^{-1}g_1(x_1...x_i) = (x_1...x_j)^{-1}g_1(x_1...x_j)$$

so

$$(x_1...x_ix_{j+1}...x_n)^{-1}g_1(x_1...x_ix_{j+1}...x_n) = g_2$$

which contradicts the minimality of x.

□

Corollary 6.4. *The conjugacy problem is solvable for hyperbolic groups.*

Proof. Indeed given $g_1, g_2 \in G$ it suffices to check whether $g_2 = xg_1x^{-1}$ for all x with

$$|x| \leq (2|S|)^{2\delta+|g_1|} + |g_1| + |g_2|$$

□

Lemma 6.3. *Let $G = \langle S \rangle$ be δ-hyperbolic for some $\delta \in \mathbb{N}$, $\delta \geq 1$. Assume that for some $g \in G$ with $|g| > 4\delta$ we have that $|g^2| \leq 2|g| - 2\delta$. Then there is some $h \in G$ conjugate to g with $|h| < |g|$.*

Proof. Consider the triangle $[1, g, g^2]$ in $\Gamma(S, G)$. By δ-thinness of this triangle we have that there are $u, s, v \in G$ such that $g = usv$ (where usv is a geodesic word), $|u| = |v| = \delta$ and $|vu| \leq \delta$. If we set $t = vu$ we have that

$$g = usv = ustu^{-1}$$

and $|st| < |g|$.

□

Lemma 6.4. *Let $G = \langle S \rangle$ be δ-hyperbolic for some $\delta \in \mathbb{N}$, $\delta \geq 1$. Assume that for some $g \in G, x \in \Gamma(S,G)$ with $d(x,gx) > 100\delta$ we have that $d(x,g^2x) > 2d(x,gx) - 8\delta$. Then*

$$d(x, g^n x) \geq nd(x, gx) - 16n\delta$$

for all $n \in \mathbb{N}$.

Proof. It suffices to show that for all n

$$d(x, g^n x) \geq d(x, g^{n-1}x) + d(x, gx) - 16\delta$$

Clearly this holds for $n = 1, 2$. We argue by induction. Assume that it is true for all $k \leq n$. We consider the triangles $[x, g^n x, g^{n+1}x]$, $[x, g^{n-1}x, g^n x]$. Assume that

$$d(x, g^{n+1}x) < d(x, g^n x) + d(x, gx) - 16\delta$$

By δ-thinness of $[x, g^n x, g^{n+1}x]$ there are vertices u_1, u_2 on the geodesics $[g^n x, g^{n+1}x]$, $[x, g^n x]$ respectively, such that

$$d(u_1, g^n x) = d(u_2, g^n x) = 5\delta, \; d(u_1, u_2) \leq \delta$$

Similarly by δ-thinness of $[x, g^{n-1}x, g^n x]$ there is a vertex $u_3 \in [g^{n-1}x, g^n x]$ such that $d(u_3, g^n x) = 5\delta$ and $d(u_2, u_3) \leq \delta$. We have then

$$d(x, g^2x) = d(g^{n-1}x, g^{n+1}x) \leq d(g^{n-1}x, u_3) + d(u_1, u_3) + d(u_1, g^{n+1}x) = 2d(x, gx) - 8\delta$$

which is a contradiction.

□

Proposition 6.4. *Let $G = \langle S \rangle$ be δ-hyperbolic for some $\delta \in \mathbb{N}$, $\delta \geq 1$. Assume that g is an element of infinite order. Then there are constants $c > 0, d \geq 0$ such that*

$$d(1, g^n) \geq cn - d$$

for all $n \in \mathbb{N}$.

Proof. It is clear that we may replace g by a power. Further it is enough to show that for some $x \in \Gamma(S, G)$ there are constants c', d' so that

$$d(x, g^n x) \geq c'n - d'$$

for all n.

In what follows we pick $n \gg k \gg 0, k, n \in \mathbb{N}$. It will be clear from the proof how k, n are chosen. We consider the geodesic $[1, g^n]$. Let m be a vertex on this geodesic at distance ≤ 1 from its midpoint. Since there are finitely many vertices in the ball $B(m, 100\delta)$ we may pick k so that

$$d(m, g^k m) \geq 100\delta$$

Now by thinness of the quadrilateral

$$[1, g^n, g^{k+n}, g^k]$$

and since $n \gg k$, we have that

$$d(g^k m, [1, g^n]) \leq 2\delta$$

In particular there is a vertex y on $[1, g^n]$ such that $d(y, g^k m) \leq 2\delta$. Then $g^k[m, y]$ is contained in the geodesic $[g^k, g^{k+n}]$ and there is some $z \in [1, g^n]$ such that $d(z, g^k y) \leq 2\delta$. It follows that

$$d(m, g^{2k} m) \geq d(m, g^k y) - 2\delta \geq 2d(m, y) - 4\delta \geq d(m, g^k m) - 8\delta$$

since $d(m, y) \geq d(m, g^k m) - 2\delta$. The assertion now follows by applying lemma 6.4 to g^k and m.

□

It follows from this proposition that if α is a geodesic from 1 to g then

$$\bigcup_n g^n \alpha$$

is a quasi-geodesic.

Proposition 6.5. *Let $G = \langle S \rangle$ be δ-hyperbolic and let $g \in G$ be an element of infinite order. Let $C(g)$ be the centralizer of g. Then the quotient $C(g)/\langle g \rangle$ is finite.*

Proof. Let $L > 0$ be such that for any $n \in \mathbb{N}$ the geodesic $[1, g^n]$ is contained in the L-neighborhood of $\{1, g, ..., g^n\}$. Let $s \in C(g)$ and $m \in \mathbb{N}$ such that

$$|g^m| \geq 2|s| + 2\delta$$

We consider the quadrilateral $[1, g^m, sg^m, s]$. By δ-thinness there is some vertex $p \in [1, g^m]$ such that

$$d(p, [s, sg^m]) \leq 2\delta$$

It follows that there are g^i, g^j such that

$$d(g^i, g^j s) \leq 2L + 2\delta$$

so

$$d(g^{i-j}, s) \leq 2L + 2\delta$$

It follows that $s = g^{i-j}u$ with $|u| \leq 2L + 2\delta$. Therefore every coset $s\langle g\rangle$ has a representative which has word length $\leq 2L + 2\delta$. Hence the quotient $C(g)/\langle g\rangle$ is finite.

□

Corollary 6.5. *If G is hyperbolic then G has no subgroup isomorphic to* $\mathbb{Z} \times \mathbb{Z}$.

6.5 More results and open problems

There is a number of results on hyperbolic groups that we were not able to present in this short introduction. We give a list of some results hoping that this will give a better perspective on the subject. Some of the results below can be proven by the techniques that we have already presented while others are quite deep requiring a quite different approach.

Theorem 6.6. *Let G be a hyperbolic group which is not finite or virtually* $\mathbb{Z}$. *Then G contains a free subgroup of rank 2.*

Theorem 6.7. *Let G be a hyperbolic group and let* $g_1, ..., g_n \in G$. *Then there is some* $N > 0$ *such that the group* $\langle g_1^N, ..., g_n^N\rangle$ *is free of rank at most n.*

Theorem 6.8. *(Gromov-Delzant) Let G be a hyperbolic group and let H be a fixed one-ended group. Then G contains at most finitely many conjugacy classes of subgroups isomorphic to H.*

Theorem 6.9. *(Sela-Guirardel-Dahmani) The isomorphism problem is solvable for hyperbolic groups.*

Theorem 6.10. *(Sela) Torsion free hyperbolic groups are Hopf.*

There is a number of open questions about hyperbolic groups:

1. Are hyperbolic groups resudually finite?
2. Let G be hyperbolic. Does G have a torsion free subgroup of finite index?
3. Gromov conjectures that if G is torsion free hyperbolic then G has finitely many torsion free finite extensions.