HYPERBOLIC SPACES

If X is a geodesic metric space, a geodesic triangle $[x, y, z]$ in X is a union of three geodesic paths $[x, y] \cup [y, z] \cup [x, z]$ where $x, y, z \in X$.

Definition 6.9. Let $\delta \geq 0$. We say that a geodesic triangle in a geodesic metric space is δ-*slim* if each side is contained in the δ-neighborhood of the two other sides. We say that a geodesic metric space X is *hyperbolic* if there is some $\delta \geq 0$ so that all geodesic triangles in X are δ-slim.

Examples. 1. Trees are hyperbolic spaces (in fact 0-hyperbolic).

2. Finite graphs are hyperbolic spaces.
3. $\mathbb{R}^2$ with the usual Euclidean metric is not hyperbolic.
4. It turns out that $\mathbb{H}^2$, the hyperbolic plane, is hyperbolic.

There are several equivalent formulations of hyperbolicity. We give one more now and we will discuss some other reformulations later in the course.

If $\Delta = [x, y, z]$ is a triangle then there is a metric tree (a 'tripod') T_Δ with 3-endpoints x', y', z' such that there is an onto map $f_\Delta : \Delta \to T_\Delta$ which restricts to an isometry from each side $[x, y], [y, z], [x, z]$ to the corresponding segments $[x', y'], [y', z'], [x', z']$. We denote by c_Δ the point $[x', y'] \cap [y', z'] \cap [x', z']$ of T_Δ.

Definition 6.10. Let $\delta \geq 0$. We say that a geodesic triangle $\Delta = [x, y, z]$ in a geodesic metric space is δ-*thin* if for every $t \in T_\Delta = [x', y', z']$, $diam(f_\Delta^{-1}(t)) \leq \delta$.

Theorem 6.4. *Let X be a geodesic metric space. The following are equivalent:*

1. *There is a $\delta \geq 0$ such that all geodesic triangles in X are δ-slim.*
2. *There is a $\delta' \geq 0$ such that all geodesic triangles in X are δ'-thin.*

Proof. Clearly 2 implies 1. Indeed one can simply take $\delta = \delta'$.

We show now that 1 implies 2. We will show that we may take $\delta' = 4\delta$.

Let $\Delta = [x, y, z]$ be a geodesic triangle and let $f_\Delta : \Delta \to T_\Delta$ the map defined above to a tripod. Let $f^{-1}(c_\Delta) = \{c_x, c_y, c_z\}$ where

$$c_x \in [y, z],\ c_y \in [x, z],\ c_z \in [x, y]$$

Let $a \in [x, c_z]$ and let a' in $[x, c_y]$ such that $d(x, a') = d(x, a)$. By symmetry it is enough to show that $d(a, a') \leq 4\delta$.

We have that

$$d(a, a_1) \leq \delta$$

for some

$$a_1 \in [x, z] \cup [y, z]$$

We distinguish two cases:

Case 1. $a_1 \in [x, z]$. Then

$$d(x, a') + \delta \geq d(x, a) + d(a, a_1) \geq d(x, a_1) \geq d(x, a) - d(a, a_1) \geq d(x, a') - \delta$$

by the triangle inequality. It follows that

$$d(a, a') \leq \delta + d(a_1, a') \leq 2\delta$$

Case 2. $a_1 \in [y, z]$. We claim that $d(a, c_x) \leq 2\delta$ in this case. Indeed if $a_1 \in [c_x, y]$ by the triangle inequality

$$d(a, y) \leq d(y, a_1) + \delta \implies d(y, a_1) \geq d(y, c_x) - \delta \implies d(a_1, c_x) \leq \delta$$

so $d(a, c_x) \leq 2\delta$.

If $a_1 \in [c_x, z]$ then again by the triangle inequality:

$$d(x, z) \leq d(x, a) + \delta + d(a_1, z) \implies d(x, z) \leq d(x, c_z) + \delta + d(a_1, z)$$

Since $d(x, z) = d(z, c_y) + d(x, c_z)$ and $d(z, c_y) = d(z, c_x)$ we obtain:

$$d(z, c_x) \leq d(a_1, z) + \delta$$

so $d(a_1, c_x) \leq \delta$ and $d(a, c_x) \leq 2\delta$. By symmetry, either, as in case 1, $d(a', a) \leq 2\delta$ or $d(a', c_x) \leq 2\delta$. It follows that

$$d(a, a') \leq 4\delta$$

Definition 6.11. Let X be a geodesic metric space. We say that X is δ-hyperbolic if all geodesic triangles in X are δ-thin.

Lemma 6.1. *Let X be a δ-hyperbolic geodesic metric space. Let $x_0, x_1, ..., x_n \in X$ and let $p \in [x_0, x_n]$. Then*

$$d(p, [x_0, x_1] \cup [x_1, x_2] ... \cup [x_{n-1}, x_n]) \leq (log_2(n) + 1)\delta$$

Proof. Let's say that $2^{k-1} < n \leq 2^k$ for $k \in \mathbb{N}$. It suffices to prove that

$$d(p, [x_0, x_1] \cup [x_1, x_2] ... \cup [x_{n-1}, x_n]) \leq k\delta.$$

We argue by induction on k. This is clearly true if $k = 1$ (ie. $n = 2$). For $k > 1$, pick $m = 2^{k-1}$. Then there is some $p_1 \in [x_0, x_m] \cup [x_m, x_n]$ with $d(p, p_1) \leq \delta$. By the inductive hypothesis

$$d(p_1, [x_0, x_1] \cup [x_1, x_2] ... \cup [x_{n-1}, x_n]) \leq (k-1)\delta$$

and the result follows.

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