GROUPS AS GEOMETRIC OBJECTS

Although geometric methods were used in group theory since its inception it was Gromov in 1984 that set the foundations of modern group theory. His insight was that one can derive many algebraic properties of infinite groups from their 'geometry'. In fact looking at the geometry turned out to be very revealing of the group structure, more so than pure algebraic manipulations. The first section of this chapter will explain what we mean by 'geometry' in this context. Riemannian geometry, even though it inspires many arguments that follow, is useless for studying finitely generated groups. Finitely generated groups are discrete objects with no interesting 'local' geometry. Their true geometry becomes apparent only from 'infinitely far away'. Gromov's insight transformed the field, as by bringing geometry into play, other tools such as analysis, dynamics etc. became available for studying groups.

One of the most convincing demonstrations of the geometric point of view is the theory of hyperbolic groups. This is a class of groups which is generic (in a precise statistical sense 'most' groups are hyperbolic) and which can be studied by geometric methods. The theory of hyperbolic groups unifies the small cancellation theory which has algebraic origin and the deep theory of negatively curved manifolds. We will show in the following sections that the word and conjugacy problem are solvable for hyperbolic groups and we will give an introduction to the geometric tools used to study them.

Quasi-isometries

We consider in the sequel connected graphs as metric spaces. So if Γ is a connected graph we identify each edge of Γ with the unit interval and the distance of any two points is defined to be the length of the shortest path joining them.

Definition 6.1. If v is a vertex of a graph Γ we define the *degree* of v to be the number of edges incident with v. So $deg(v) = card\{e \in E(\Gamma) : o(e) = v\}$. We say that a graph Γ is *locally finite* if every vertex is incident to finitely many edges. A graph is called *regular* if all vertices have the same degree. A subgraph L of Γ is a *bi-infinite geodesic* if it is isometric to $\mathbb{R}$ (where we consider L to be equipped with the metric induced by Γ).

We remark that if Γ is the Cayley graph of a finitely generated group then Γ is a regular locally finite graph.

We recall the definition of the Cayley graph of a group:

Definition 6.2. Let G be a group generated by a finite set S. The Cayley graph of G, $\Gamma = \Gamma(S, G)$, is the graph with vertex set

$$V = \{g : g \in G\}$$

and edge set

$$E = \{(g, gs), g \in G, s \in S\}$$

We can see G as a subset of Γ, so the metric of Γ induces a metric d_S on G, called the *word metric* of G. We remark that

$$d_S(g, e) = \min\{n : g = s_1^{\pm 1} ... s_n^{\pm 1}, s_1, ..., s_n \in S\}$$

In this way we can associate to a finitely generated group G a metric space or view G itself as a metric space. There is a problem however, the graph we defined depends on the generating set S. In general given a group G there is no natural way to pick a generating set S and different generating sets give different graphs (and word metrics) for G!

Example 6.1. Consider the Cayley graphs of $\mathbb{Z}$ equipped with 2 different generating sets: $S_1 = \{1\}$, $S_2 = \{2, 3\}$.

One sees from this example that Cayley graphs for the same group can look completely different. One may remark however that when viewed from 'far away' these graphs look similar. Although the 'local geometry' of Cayley graphs changes when we change generating sets the 'large scale' geometry is preserved.

We make this remark precise by introducing quasi-isometries.

Definition 6.3. A (usually non-continuous) map between metric spaces $f : X \to Y$ is called a *quasi-isometry* if there exist $K \geq 1, A > 0$ such that

- for all $x_1, x_2 \in X$

$$\frac{1}{K}d(x_1,x_2) - A \leq d(f(x_1), f(x_2)) \leq Kd(x_1,x_2) + A, \quad and$$

- for all $y \in Y$ there is some $x \in X$ such that $d(y, f(x)) \leq A$.

When there is a quasi-isometry $f : X \to Y$ we say that X, Y are quasi-isometric and we write $X \sim Y$.

Example 6.2. 1. $\mathbb{R}$ and $\mathbb{Z}$ are quasi-isometric.

2. Any metric space of finite diameter is quasi-isometric to a point.

Exercises 6.1. 1. Show that $\sim$ is an equivalence relation.

2. Let S_1, S_2 be finite generating sets of a group G. Show that $\Gamma(S_1, G) \sim \Gamma(S_2, G)$.

3. Let T_3, T_4 be the regular trees of degrees, respectively, 3,4. Show that T_3, T_4 are quasi-isometric.

We remark that if a group G is not finitely generated we can not associate a 'geometry' to the group in this way. Indeed if we take as generating set the set of all elements of G the Cayley graph is a bounded metric space, so it is quasi-isometric to a point.

Given $\epsilon, \delta > 0$ a subset N of a metric space X is called an (ϵ, δ)-*net* (or simply a net) if for every $x \in X$ there is some $n \in N$ such that $d(x, n) \leq \epsilon$ and for every $n_1, n_2 \in N$, $d(n_1, n_2) \geq \delta$.

A set N that satisfies only the second condition (i.e. for every $n_1, n_2 \in N$, $d(n_1, n_2) \geq \delta$) is called δ-separated.

Exercises 6.2. 1. Show that any metric space X has a $(1,1)$-net.

2. Show that if $N \subset X$ is a net then $X \sim N$.

3. Show that $X \sim Y$ if and only if there are nets $N_1 \subset X, N_2 \subset Y$ and a bilipschitz map $f : N_1 \to N_2$.

4. Give an example of a metric space which is not quasi-isometric to any graph.

5. Let G be a finitely generated group. Show that $H < G$ is a net in G if and only if H is a finite index subgroup of G.

It turns out that if a finitely generated group acts 'nicely' on a 'nice' metric space then the space is quasi-isometric to the group.

We make this precise below.

Definition 6.4. Let $p : [0,1] \to X$ be a path in a metric space (X, d). We define the *length* of p to be the supremum of

$$\sum_{i=0}^{n} d(p(t_i), p(t_{i+1}))$$

over all partitions $0 = t_0 < t_1 < ... < t_n = 1$ $(n \in \mathbb{N})$ of $[0,1]$. We say that X is a *geodesic metric space* if for any $a, b \in X$ there is a path p joining a, b such that $length(p) = d(a, b)$. Such a path p is called *geodesic*.

It will be convenient to parametrize paths with respect to arc-length. We recall that a path $p : [0, l] \to X$ is said to be parametrized by arc-length if

$$|t - s| = length(p([t, s])), \ \forall t, s \in [a, b]$$

If X is a geodesic metric space and $a, b \in X$ we denote by $[a, b]$ a geodesic path joining them.

Examples. 1. Connected graphs with the metric defined earlier are geodesic metric spaces.

2. $\mathbb{R}^n$ with the Euclidean distance and, more generally, complete Riemannian manifolds are geodesic metric spaces (Hopf-Rinow).

3. $\mathbb{R}^2 - \{(0,0)\}$ is not a geodesic metric space.

Definition 6.5. We say that a metric space X is *proper* if every closed ball in X is compact.

Example 6.3. A graph with a vertex of infinite degree is not a proper metric space.

Definition 6.6. Assume that a group G acts on a metric space X by isometries. We say that the action is *co-compact* if there is a compact $K \subset X$ such that

$$\bigcup_{g \in G} \{gK\} = X$$

We say that G acts *properly discontinuously* on X if for every compact $K \subset X$ the set $\{g \in G : gK \cap K \neq \emptyset\}$ is finite.

Theorem 6.1. *(Milnor-Švarč lemma) Let X be a proper geodesic metric space. If G acts by isometries, properly discontinuously and co-compactly on X then:*

1) G is finitely generated.

2) If S is a finite generating set of G the map

$$f : \Gamma(S, G) \to X, \ \ g \mapsto gx_0$$

is a quasi-isometry (for any fixed $x_0 \in X$).

Proof. Let $R > 0$ be such that the G-translates of $B = B(x_0, R)$ cover X, i.e.

$$\bigcup_{g\in G}\{gB\} = X$$

The set

$$S = \{s \in G : d(sx_0, x_0) \leq 2R + 1\}$$

is finite since the action of G is properly discontinuous. We claim that S is a generating set of G. Indeed let $g \in G$. Consider a geodesic path $[x_0, gx_0]$. If

$$k - 1 < d(x_0, gx_0) \leq k, \ (k \in \mathbb{N})$$

consider $x_1, ..., x_k = gx_0$ such that $d(x_i, x_{i+1}) \leq 1$ for all $i = 0, ..., k-1$. Pick $g_i \in G$, $i = 1, ..., k-1$ such that $d(g_i x_0, x_i) \leq R$. Then $d(g_i x_0, g_{i+1}x_0) \leq 2R + 1$ so $g_i^{-1} g_{i+1} \in S$. We pick $g_0 = e, g_k = g$. We have then

$$g = g_k = (eg_1)(g_1^{-1}g_2)...(g_{k-2}^{-1}g_{k-1})(g_{k-1}^{-1}g_k)$$

So g can be written as a product of elements in S.

Let's denote now by d_S the distance in $\Gamma(S, G)$. The previous calculation shows that

$$d(gx_0, x_0) \geq d_S(g, e) - 1$$

Assume that $d_S(g, e) = n$, so $g = s_1...s_n$ where $s_i \in S \cup S^{-1}$ for all i. Then

$$d(gx_0, x_0) = d(s_1...s_n x_0, x_0) \leq d(s_1...s_n x_0, s_1...s_{n-1}x_0) + ... + d(s_1 x_0, x_0) \leq (2R + 1)n$$

So

$$d(gx_0, x_0) \leq (2R + 1)d_S(g, e)$$

It follows that the map $g \to gx_0$ is a quasi-isometry between $\Gamma(S, G)$ and Gx_0.

Since S is finite the set $S' = \{g \in S : gx_0 \neq x_0\}$ is finite. Let

$$r = \min\{d(gx_0, x_0) : g \in S'\}$$

We remark that $N = \{gx_0 : g \in G\}$ is an (R, r)-net of X, so the identity map $i : N \to X$, $i(gx_0) = gx_0$ is a quasi-isometry, so $f = f \circ i$ is a quasi-isometry from G to X. □

Corollary 6.1. *1. Let $G =< S >$ be a finitely generated group and let H be a finite index subgroup of G. Then H is quasi-isometric to G.*

2. Let G be a finitely generated group and let N be a finite normal subgroup of G. Then G/N is quasi-isometric to G.

Proof. 1. H acts freely and co-compactly on $\Gamma(S,G)$.

2. G acts properly discontinuously and co-compactly on the Cayley graph of G/N. □

In geometric group theory we 'identify' groups which differ by a 'finite amount' as in the corollary above.

We give now some examples of algebraic properties that are preserved by quasi-isometries.

Exercise 6.1. Let $G =< S|R >$ be a finitely presented group and let H be a finitely generated group quasi-isometric to G. Then H is finitely presented.

Definition 6.7. If $G =< S >$ is a finitely generated group we define the growth function of G to be

$$vol_{S,G}(r) = |B(r)|$$

where $B(r)$ is the ball of radius r in (G, d_S) centered at e.

We define an equivalence relation on functions $f : \mathbb{R}^+ \to \mathbb{R}^+$. We say that $f \prec g$ if there are $A, B, C > 0$ such that for all $r \in \mathbb{R}^+$ we have $f(r) \leq Ag(Br) + C$. We note that $\prec$ is a partial order.

We say that $f \sim g$ if $f \prec g$ and $g \prec f$. $\sim$ is clearly an equivalence relation.

Exercise 6.2. Show that if $G_1 =< S >, G_2 =< S' >$ are finitely generated quasi-isometric groups then $vol_{S,G_1} \sim vol_{S',G_2}$. Deduce that the growth function of a group does not depend (up to equivalence) on the generating set that we pick.

Usually one considers this function up to equivalence, and denotes it by $vol_G(r)$.

Theorem 6.2. *(Gromov) A finitely generated group G has a nilpotent subgroup of finite index if and only if $vol_G(r) \prec r^n$ for some $n \in \mathbb{N}$.*

It follows from this theorem that if G is quasi-isometric to a finitely generated nilpotent group then G has a nilpotent subgroup of finite index.

Definition 6.8. (ends) Let Γ be a locally finite graph. If $K \subset \Gamma$ is compact we define $c(K)$ to be the number of unbounded connected components of $\Gamma - K$. We define then the *number of ends* of Γ to be

$$e(\Gamma) = \sup\{c(K) : K \subset \Gamma, \ \ compact\}$$

We remark that we obtain an equivalent definition if, instead of compact sets K, we consider finite sets of vertices of Γ. Clearly finite graphs have 0 ends.

For a finitely generated group G we define the number of ends, $e(G)$, of G to be the number of ends of the Cayley graph of G.

Exercise 6.3. Show that two quasi-isometric locally finite graphs have the same number of ends. Deduce that the number of ends of a finitely generated group is well defined (ie it does not depend on the Cayley graph that we pick).

Exercise 6.4. Show that a finitely generated group has 0,1,2 or ∞ ends.

For example $\mathbb{Z}^2$ has 1 end, $\mathbb{Z}$ has 2 ends while $\mathbb{F}_2$ has ∞ ends.

It turns out that the number of ends of the Cayley graph of a group tells us whether the group splits over a finite group:

Theorem 6.3. *(Stallings) A finitely generated group G splits over a finite group if and only if G has more than 1 end.*

It is easy to see (exercise) that if a f.g. group G splits over a finite group then $e(G) > 1$. So the interesting direction of the theorem is: if $e(G) > 1$ then G splits over a finite group.

Stallings theorem combined with Dunwoody's accessibility theorem implies that if a finitely generated group is quasi-isometric to a free group then it has a finite index subgroup which is free.

We treat now the easier case of groups quasi-isometric to $\mathbb{Z}$.

Proposition 6.1. *Let G be a finitely generated 2-ended group. Then G has a finite index subgroup isomorphic to $\mathbb{Z}$.*

Proof. Let Γ be the Cayley graph of G. We consider a compact connected set K such that $\Gamma - K$ has 2 unbounded connected components C, D.

We claim that there is some $a \in G$ such that aC is properly contained in C. Indeed pick g such that gK is contained in C. Then at least one unbounded component of $\Gamma - gK$ does not contain K. If this is gC, then, since gC is connected, gC is properly contained in C and we are done. Otherwise gD is contained in C. Pick now h such that hK is contained in gD. If hD is properly contained in gD then $g^{-1}hD$ is properly contained in D. Set $a = g^{-1}h$ and rename D to C. Otherwise hC is properly contained in gD, hence it is properly contained in C.

We remark that $aC \subset C$ and $aC \neq C$. So $a^2C \subset aC \subset C$. Inductively we have $a^nC \subset C$, $a^nC \neq C$. It follows that a is an element of infinite order.

We note now that $K \cap aK = \emptyset$. Since $d(e, a^n) \to \infty$ for any vertex $v \in \Gamma$, there is some $n \in \mathbb{Z}$ such that v is either contained in $a^n K$ or v is contained in a bounded component of $\Gamma - (a^{n-1}K \cup a^n K)$.

It follows that

$$\{a^n : n \in \mathbb{Z}\}$$

is a net in Γ. So $< a >$ is a finite index subgroup of G. □

Corollary 6.2. *Let G be a finitely generated group quasi-isometric to $\mathbb{Z}$. Then G has a finite index subgroup isomorphic to $\mathbb{Z}$.*