GRAPHS OF GROUPS AND ACTIONS ON TREES

Let (G, Y) be a graph of groups. We will show in this section that the fundamental group of this graph of groups acts on a tree T so that the quotient graph of this action is isomorphic to Y.

The construction of T resembles the construction of the universal cover in topology. The universal cover $\tilde{X}$ of a space X is defined using the paths of X modulo an equivalence relation (homotopy). Here we do something similar: we consider paths in the graph of groups. The group elements on the paths account for the branching of the tree. A trivial case which illustrates this point is the case of a $\mathbb{Z}_2$ action on a tree with 2 edges fixing the vertex in the middle and permuting the 2 edges. The quotient space is just a single edge, so topologically it is the universal cover of itself. However we can recover the original 2-edge tree using the $\mathbb{Z}_2$ stabilizer of the middle vertex.

Let $a_0 \in Y$. We consider the set of paths in (G, Y):

$$\pi[a_0, a] = \{|c| : c \text{ path from } a_0 \text{ to } a\}$$

We define an equivalence relation in $\pi[a_0, a]$: $|c_1| \sim |c_2|$ if $|c_1| = |c_2|g$ for some $g \in G_a$. We define then

$$V(T) = \bigcup_{a \in V(Y)} \pi[a_0, a]/\sim$$

We remark that an element of $\pi[a_0, a]/\sim$ corresponds to a unique S-reduced path of the form: $(s_1, e_1, ..., s_n, e_n)$ where $t(e_n) = a$ and $o(e_1) = a_0$. Indeed note that

$$|(s_1, e_1, ..., s_n, e_n)| \sim |(s_1, e_1, ..., s_n, e_n, g)| \;\; (g \in G_a)$$

So we may identify the vertices of T with S-reduced paths of the form $(s_1, e_1, ..., s_n, e_n)$. An edge of T now is given by a pair of S-reduced paths that differ by an edge of Y:

$$\{(s_1, e_1, ..., s_n, e_n),\; (s_1, e_1, ..., s_n, e_n, s_{n+1}, e_{n+1})\}$$

Clearly T is connected since (1) can be joined to any other vertex by a path. Moreover, it follows from lemma 5.1 that if $v \in V(T)$ there is a unique S-reduced path joining $(1), v$. Therefore T is a tree.

We define now the action of $H = \pi_1(G, Y, a_0) = \pi[a_0, a_0]$ on T. If $g \in \pi[a_0, a_0]$ and $v \in \pi[a_0, a]$ then $gv \in \pi[a_0, a]$. So we define $g \cdot [v] = [gv]$ (where we denote by $[v]$ the equivalence class of v in $\pi[a_0, a]/\sim$). This defines an action of H on $V(T)$ since $(g_1 g_2) \cdot [v] = g_1 \cdot (g_2 \cdot [v])$. We note that adjacent vertices go to adjacent vertices under this action so we have an action on T. We remark that if $v_1, v_2 \in \pi[a_0, a]$ then

$v_2v_1^{-1} \in \pi[a_0, a_0]$ and $(v_2v_1^{-1})\cdot[v_1] = [v_2]$. It follows that we can identify the vertices of the quotient graph T/H with the vertices of Y. We show now that the edges of the quotient graph T/H correspond to the edges of Y too. Let $e_1 = ([v], [vs_1e]), e_2 = ([v], [vs_2e])$ be two edges of T with $o(e_1) = o(e_2) = [v]$, $s_1, s_2 \in G_{o(e_1)}$. If $g = v(s_2s_1^{-1})v^{-1}$ we have that $g \in \pi[a_0, a_0]$ and $g \cdot e_1 = e_2$. So both edges lie in the same orbit and this orbit corresponds to the edge $e \in E(Y)$.

We can see further that stabilizers of vertices and edges of T are conjugates of vertex and edge groups of (G, Y). Precisely:

Proposition 5.3. *1. If $[v] \in V(T)$ and $v \in \pi[a_0, b]$ then $stab([v]) = vG_bv^{-1}$.*
2. If $\delta \in E(T)$, $\delta = [[v], [vge]]$ where $e = [a, b]$, $g \in G_a$ then $stab(\delta) = (vg)(\alpha_{\bar{e}}(G_e)(vg)^{-1}$.

Proof. 1. Clearly $vG_bv^{-1} \subset stab([v])$. Assume now that $g \in stab([v])$. Then by the definition of $V(T)$ $gv = vg_b$, $g_b \in G_b$. So $g \in vG_bv^{-1}$. We conclude that $stab([v]) = vG_bv^{-1}$.

2. $stab(\delta) = stab([v]) \cap stab([vge])$. So

$$stab(\delta) = vG_av^{-1} \cap (vge)G_b(vge)^{-1} = v(G_a \cap geG_be^{-1}g^{-1})v^{-1} =$$

$$= (vg)(G_a \cap eG_be^{-1})(vg)^{-1}$$

since $g \in G_a$. We remark that $eG_be^{-1} \cap G_a = \alpha_{\bar{e}}(G_e)$. This is because if $g_b \in G_b$, either eg_be^{-1} is a reduced word and so does not lie in G_a or $g_b \in \alpha_e(G_e)$ and then $eg_be^{-1} \in G_a$. We conclude that

$$stab(\delta) = (vg)(\alpha_{\bar{e}}(G_e)(vg)^{-1}$$

□

We denote the tree T by $(\widetilde{G, Y, a_0})$ and we say that it is the *universal covering tree* of the graph of groups (G, Y).

5.4 Quotient graphs of groups

We showed in the previous section that if $\pi_1(G, Y, a_0)$ is the fundamental group of a graph of groups then $\pi_1(G, Y, a_0)$ acts on a tree T with quotient graph Y. The converse is also true: If a group Γ acts on a tree T with quotient Y, then there is a graph of groups (G, Y) so that $\pi_1(G, Y, a_0) = \Gamma$.

We explain now how to associate a graph of groups (G, Y) to an action $\Gamma \curvearrowright T$ (where T is a tree). We define $Y = T/\Gamma$. We have the projection map $p : T \to Y$. Let $X \subset S \subset T$ be subtrees of T such that $p(X)$ is a maximal tree of Y, $p(S) = Y$ and the map p restricted to S is 1-1 on the set of edges. We introduce some convenient notation: if v, e are respectively a vertex and an edge of Y we write v^X for the vertex of X for which $p(v^X) = v$ and e^S for the edge of S for which $p(e^S) = e$. We define now a graph of groups with Y as underlying graph. If $v \in V(Y)$ we set $G_v = stab(v^X)$. If $e \in E(Y)$ we set $G_e = stab(e^S)$. It remains to define monomorphisms $\alpha_e : G_e \to G_{t(e)}$. For every $x \in V(S)$ we pick $g_x \in \Gamma$ such that $g_x x \in X$. If $x \in X$ we take $g_x = 1$. If $x = t(e^S)$ we define:

$$\alpha_e : G_e \to G_{t(e)}, \text{ by } \alpha_e(g) = g_x g g_x^{-1}$$

In this way we define a graph of groups (G, Y). We define a homomorphism $\varphi : F(G, Y) \to \Gamma$ as follows: $\varphi|_{G_a} = id$ for all $a \in V(Y)$. If $e \in E(Y)$ and $y = o(e^S)$, $x = t(e^S)$ then we define $\varphi(e) = g_y g_x^{-1}$. We verify that the relations are satisfied:

$$\varphi(e\alpha_e(g)e^{-1}) = (g_y g_x^{-1})(g_x g g_x^{-1})(g_y g_x^{-1})^{-1} = g_y g g_y^{-1}$$

and

$$\varphi(\alpha_{\bar{e}}(g)) = g_y g g_y^{-1}$$

So φ is indeed a homomorphism. We note that if $e \in p(X)$ then $\varphi(e) = 1$ so we have in fact a homomorphism

$$\varphi : \pi_1(G, Y, p(X)) = \pi_1(G, Y, a_0) \to \Gamma$$

We have the following:

Theorem 5.2. *The map $\varphi := \pi_1(G, Y, a_0) \to \Gamma$ is an isomorphism. If $\tilde{T}$ is the universal covering tree of (G, Y) then there is a graph morphism $\psi : \tilde{T} \to T$ such that ψ is 1-1 and onto and $\psi(gv) = \varphi(g)\psi(v)$ for all $v \in V(\tilde{T})$, $g \in \pi_1(G, Y, a_0)$.*

We omit the proof of this theorem. What this theorem essentially says is that we can recover the group and the action on the tree by the quotient graph of groups.

We can now understand subgroups of fundamental groups of graphs of groups.

Theorem 5.3. *Let $\Gamma = \pi_1(G, Y, a_0)$ where (G, Y) is a graph of groups. If B is a subgroup of Γ then there is a graph of groups (H, Z) such that $B = \pi_1(H, Z, b_0)$ and for every $v \in V(Z)$, $e \in E(Z)$, $H_v \leq gG_a g^{-1}$, $H_e \leq \gamma G_y \gamma^{-1}$ for some $a \in V(Y)$, $y \in E(Y)$ and $g, \gamma \in \Gamma$.*

Proof. Γ acts on a tree T with quotient graph of groups (G, Y). Since $B \leq \Gamma$, B acts also on T and the vertex and edge stabilizers of B are contained in the vertex and edge stabilizers of Γ. If $Z = T/B$ it is clear that the quotient graph of groups (H, Z) that we obtain from the action of B satisfy the assertions of the theorem. □

Corollary 5.3. *(Kurosh's theorem) Let* $G = G_1 * ... * G_n$*. If* $H \leq G$ *then* $H = (\underset{i \in I}{*} H_i) * F$ *where* F *is a free group and the* H_i*'s are subgroups of conjugates of the* G_j*'s.*

Proof. G is the fundamental group of a graph of groups with underlying graph a tree with n vertices labeled by $G_1, ..., G_n$ and trivial edge groups. We apply now the previous theorem. □

We mention two important theorems on the structure of finitely presented groups.

We say that a group G is *indecomposable* if it can not be written as a non-trivial free product $G = A * B$.

Theorem 5.4. *(Grushko) Let* G *be a finitely generated group. There are finitely many indecomposable groups* $G_1, ..., G_k$ *and* $n \geq 0$ *such that*

$$G = G_1 * ... * G_k * \mathbb{F}_n$$

Moreover if we have another decomposition of G *as*

$$G = H_1 * ... * H_m * \mathbb{F}_r$$

where H_i *are indecomposable then* $m = k$, $r = n$, *and after reordering* H_i *is conjugate to* G_i *for all* i.

Theorem 5.5. *(Dunwoody) Let* Γ *be a finitely presented group. Then* Γ *can be written as* $\Gamma = \pi_1(G, Y, a_0)$ *where* (G, Y) *is a finite graph of groups such that all edge groups are finite and all vertex groups do not split over finite groups.*

Dunwoody has shown that this last theorem does not generalize to all finitely generated groups.