GRAPHS OF GROUPS

Fundamental groups of graphs of groups

Definition 5.1. A *graph of groups* (G,Y) consists of a connected graph Y and a map G such that

1. G assigns a group G_v to every vertex $v \in V(Y)$ and a group G_e to every edge $e \in E(Y)$, so that $G_e = G_{\bar{e}}$.
2. For each edge group G_e there is a monomorphism $\alpha_e : G_e \to G_{t(e)}$.

Graphs of groups occur naturally in the context of group actions on trees. If a group G acts on a tree T without inversions then we can form the quotient graph $Y = T/G$. We note that there is a projection $p : T \to T/G$.

To each vertex $v \in Y$ (or edge $e \in Y$) we associate a group G_v (G_e) where G_v is the stabilizer of a vertex in $p^{-1}(v)$ (edge in $p^{-1}(e)$). Note that all stabilizers of vertices in $p^{-1}(v)$ are isomorphic and the same holds for edges. If the vertex $v' \in p^{-1}(v)$ is an endpoint of the edge $e' \in p^{-1}(e)$ in T we have a monomorphism (inclusion) $stab(e') \to stab(v')$ and this is how we obtain the monomorphism $G_e \to G_v$. We will associate graphs of groups to actions more formally later, here we mention this as a source of examples and in order to put this definition in context.

Definition 5.2. The *path group* of the graph of groups (G,Y) is the group

$$F(G,Y) = \langle \mathop{*}_{v \in V(Y)} G_v \mathop{*}_{e \in E(Y)} \langle e \rangle \mid \bar{e} = e^{-1},\ e\alpha_e(g)e^{-1} = \alpha_{\bar{e}}(g),\ \forall e \in E(Y),\ g \in G_e \rangle$$

If $G_v = \langle S_v \mid R_v \rangle$ then a presentation of $F(G,Y)$ is given by

$$\langle \bigcup_{v \in V(Y)} S_v \cup \{e \in E(Y)\} \mid \bigcup_{v \in V(Y)} R_v,\ \bar{e} = e^{-1},\ e\alpha_e(g)e^{-1} = \alpha_{\bar{e}}(g),\ \forall e \in E(Y),\ v \in V(Y),\ g \in G_e \rangle$$

Remarks.

1. If $G_v = \{1\}$ for all $v \in V(Y)$ then $F(G,Y) = F(E^+(Y))$ (the free group with basis the geometric edges of Y).

2. If $G_e = \{1\}$ for all $e \in E(Y)$ then $F(G,Y) = \underset{v \in V(Y)}{*} G_v * F(E^+(Y))$.

3. There is an epimorphism $F(G,Y) \to F(E^+(Y))$ defined by sending all $g \in G_v$ (for all v) to 1.

Definition 5.3. A *path* c in the graph of groups (G,Y) is a sequence

$$c = (g_0, e_1, g_1, e_2, ..., g_{n-1}, e_n, g_n)$$

such that $t(e_i) = o(e_{i+1})$ and $g_i \in G_{o(e_{i+1})} = G_{t(e_i)}$ for all i. If

$$v_0 = o(e_1), v_1 = o(e_2) = t(e_1), ..., v_n = t(e_n)$$

we say that c is a path from v_0 to v_n and $(v_0, ..., v_n)$ is the sequence of vertices of the path c. We define $|c|$ to be the element of the path group: $|c| = g_0 e_1 g_1 .... e_n g_n$.

If $a_0, a_1 \in V(Y)$ we define

$$\pi[a_0, a_1] = \{|c| : c \text{ path from } a_0 \text{ to } a_1\}$$

If $a_0, a_1, a_2 \in V(Y)$ and $\gamma \in \pi[a_0, a_1]$, $\delta \in \pi[a_1, a_2]$ then $\gamma \cdot \delta \in \pi[a_0, a_2]$.

Proposition 5.1. *Let (G,Y) be a graph of groups. The set $\pi[a_0, a_0]$ ($a_0 \in V(Y)$) is a subgroup of $F(G,Y)$. We call this fundamental group of the graph of groups (G,Y) with base point a_0 and we denote it by $\pi_1(G,Y,a_0)$.*

Proof. It is enough to show that every element of $\pi[a_0, a_0]$ has an inverse in $\pi[a_0, a_0]$. If $c = (g_0, e_1, g_1, e_2, ..., g_{n-1}, e_n, g_n)$ is a path from a_0 to a_0 then

$$|c|^{-1} = g_n^{-1} \bar{e}_n .... \bar{e}_1 g_0^{-1} \in \pi[a_0, a_0]$$

□

Definition 5.4. Let (G,Y) be a graph of groups and let T be a maximal tree of Y. We define the *fundamental group* of (G,Y) with respect to T, $\pi_1(G,Y,T)$ to be the quotient group

$$\pi_1(G,Y,T) = F(G,Y)/\langle\langle \{e, e \in T\} \rangle\rangle$$

We have the obvious quotient map $q : F(G,Y) \to \pi_1(G,Y,T)$.

Proposition 5.2. *The restriction of q to $\pi_1(G,Y,a_0)$ is an isomorphism, so*

$$\pi_1(G,Y,a_0) \cong \pi_1(G,Y,T)$$

Proof. We would like to define a homomorphism $f : \pi_1(G,Y,T) \to \pi_1(G,Y,a_0)$. Let $a \in V(Y)$ and $(e_1, ..., e_n)$ a geodesic path on T from a_0 to a. We set $g_a = e_1...e_n \in F(G,Y)$. If $a = a_0$ we set $g_a = 1$.

If e is an edge with $o(e) = a$, $t(e) = b$ we define

$$f(e) = g_a e g_b^{-1} \in \pi_1(G,Y,a_0)$$

Clearly if $e \in T$ then $f(e) = 1$ so this makes sense.

If $g \in G_a$ we define

$$f(g) = g_a g g_a^{-1} \in \pi_1(G,Y,a_0).$$

If e is an edge and $o(e) = P$, $t(e) = Q$ then

$$f(e\alpha_e(g)e^{-1}) = (g_P e g_Q^{-1})(g_Q \alpha_e(g) g_Q^{-1})(g_Q e g_P^{-1}) = g_P e \alpha_e(g) e^{-1} g_P^{-1} = g_P \alpha_{\bar{e}}(g) g_P^{-1}$$

and

$$f(\alpha_{\bar{e}}(g)) = g_P \alpha_{\bar{e}}(g) g_P^{-1}$$

so the relations are satisfied for all $e \in E(Y)$. It follows that f is a homomorphism.

Also $q \circ f(g) = g$ for all $g \in G_v$, $v \in V(T)$ and $q \circ f(e) = e$ for all $e \notin T$. So $q \circ f = id$.

We calculate now $f \circ q$. Let $(g_0, e_1, ..., e_n, g_n)$ be a path such that $g_0, g_n \in G_{a_0}$. If $e_i = [P_{i-1}, P_i]$ then $q(g_i) = g_i$ and $f(g_i) = g_{P_i} g_i g_{P_i}^{-1}$. Also $q(e_i) = e_i$ and $f(e_i) = g_{P_{i-1}} e_i g_{P_i}^{-1}$. We remark also that $g_{P_0} = g_{P_n} = g_{a_0} = 1$.

So

$$f \circ q(g_0 e_1 ... e_n g_n) = g_0(e_1 g_{P_1}^{-1}) g_{P_1} ... g_{P_{n-1}}^{-1}(g_{P_{n-1}} e_n g_n) = g_0 e_1 ... e_n g_n$$

so $f \circ q = id$. □

Corollary 5.1. *The fundamental group of the graph of groups $\pi_1(G,Y,a_0)$ does not depend on the basepoint a_0.*

5.2 Reduced words

Definition 5.5. Let (G,Y) be a graph of groups and let $c = (g_0, e_1, g_1, e_2, ..., g_{n-1}, e_n, g_n)$ be a path. We say that c is *reduced* if:

1) $g_0 \neq 1$ if $n = 0$.

2) For every i if $e_{i+1} = \bar{e}_i$ then $g_i \notin \alpha_{e_i}(G_{e_i})$.

If c is a reduced path we say that $g_0 e_1 .... e_n g_n$ is a reduced word. We denote by $|c|$ the element of $F(G,Y)$ represented by the word $g_0 e_1 .... e_n g_n$.

Theorem 5.1. *If c is a reduced path then $|c| \neq 1$ in $F(G,Y)$. In particular for any vertex $v \in V(Y)$ the homomorphism $G_v \to F(G,Y)$ is injective.*

Proof. We prove first the theorem for finite graphs by induction on the number of edges. If Y is a single vertex there is nothing to prove. Otherwise we distinguish two cases:

Case 1: $Y = Y' \cup \{e\}$ where Y' is a connected graph and $v = t(e) \notin Y'$. In this case

$$F(G,Y) = (F(G,Y') * G_v) \underset{\alpha_e(G_e)}{*}$$

and a reduced word on $F(G,Y)$ corresponds to a reduced word in the HNN extension which is non trivial by corollary 4.7.

Case 2: $Y = Y' \cup \{e\}$ where Y' is a connected graph and $o(e), t(e) \in Y'$. In this case

$$F(G,Y) = F(G,Y') \underset{\alpha_e(G_e)}{*}$$

and a reduced word on $F(G,Y)$ corresponds to a word in the HNN extension which is non trivial by corollary 4.7.

This proves the theorem in case Y is finite. If Y is infinite and a reduced word w is equal to 1 in $F(G,Y)$ then it is equal to a product of finitely many conjugates of relators of $F(G,Y)$. However these relators involve only group elements and edge generators lying in a finite subgraph Y_1. By taking Y_1 big enough we may assume that the conjugating elements also lie in Y_1. It follows that $w = 1$ in $F(G,Y_1)$ which is a contradiction since w is a reduced word and Y_1 is finite.

□

Corollary 5.2. *For any vertex $v \in V(Y)$ the homomorphism $G_v \to \pi_1(G,Y,T)$ is injective.*

Proof. The homomorphism $G_v \to \pi_1(G,Y,v)$ is injective since $\pi_1(G,Y,v)$ is a subgroup of $F(G,Y)$ and if $1 \neq g \in G_v$ g is a reduced word in $F(G,Y)$ hence $g \neq 1$. However $\pi_1(G,Y,v) \cong \pi_1(G,Y,T)$ and $g \in G_v$ maps to itself in $\pi_1(G,Y,T)$ so $g \neq 1$ in $\pi_1(G,Y,T)$.

□

Remark 5.1. If Y consists of a single edge $e = [u,v]$ with $u \neq v$ then one sees from the presentation that $\pi_1(G,Y,T) = G_u \underset{G_e}{*} G_v$. If the endpoints of e are equal $(u = v)$ then

$\pi_1(G, Y, T) = G_v \underset{\alpha_e(G_e)}{*}$ where the homomorphism of the HNN extension $\theta : \alpha_e(G_e) \to G_v$ is given by $\theta(g) = \alpha_{\bar{e}} \circ \alpha_e^{-1}$ and the stable letter is e.

In general if $Y = Y' \cup e$ and $t(v) \notin Y'$ then

$$\pi_1(G, Y, T) = \pi_1(G, Y', T') \underset{G_e}{*} G_v$$

while if $t(v) \in Y'$ then

$$\pi_1(G, Y, T) = \pi_1(G, Y', T) \underset{\alpha_e(G_e)}{*}$$

As we did for amalgams and HNN-extensions we can find a set of words that is in one to one correspondence with the elements of the fundamental group of the graph of groups.

Let (G, Y) be a graph of groups. For each edge $e \in E(Y)$ we pick a set S_e of left coset representatives of $\alpha_{\bar{e}}(G_e)$ in $G_{o(e)}$. We require that $1 \in S_e$.

Definition 5.6. We say that the path $(s_1, e_1, ..., s_n, e_n, g)$ is *S-reduced* if $s_i \in S_{e_i}$ for all i and $s_i \neq 1$ if $e_{i-1} = \bar{e}_i$.

Lemma 5.1. *Let $a, b \in V(Y)$. Then every element of $\pi[a, b]$ is represented by a unique S-reduced path.*

Proof. Existence. For every element $\gamma \in \pi[a, b]$ there is a reduced path $c = (g_1, e_1, g_2, e_2, ..., g_n, e_n, g)$ such that $\gamma = |c|$. We can write $g_1 = s_1 h_1,\ s_1 \in S_{e_1},\ h_1 \in \alpha_{\bar{e}_1}(G_{e_1})$. So

$$g_1 e_1 = s_1 h_1 e_1 = s_1 e_1 \bar{e}_1 h_1 e_1 = s_1 e_1 \alpha_{e_1}(h_1)$$

So we replace c by $(s_1, e_1, \alpha_{e_1}(h_1) g_2, e_2, ..., e_n, g_n)$ and we continue similarly replacing $\alpha_{e_1}(h_1) g_2$ and so on till we arrive at an S-reduced path c' such that $|c'| = \gamma$.

Uniqueness. Let

$$c = (s_1, e_1, ..., s_n, e_n, g),\ c' = (t_1, y_1, ..., t_k, y_k, h)$$

be S-reduced paths such that $|c| = |c'|$. Then

$$s_1 e_1 .... s_n e_n g = t_1 y_1 .... t_k y_k h \Rightarrow h^{-1} y_k^{-1} ... y_1^{-1} t_1^{-1} s_1 e_1 .... s_n e_n g = 1$$

Obviously this word is not reduced so $y_1 = e_1$ and $t_1^{-1} s_1 \in \alpha_{\bar{e}_1}(G_{e_1})$. Since t_1, s_1 are left coset representatives of $\alpha_{\bar{e}_1}(G_{e_1})$ we have $t_1 = s_1$. So $y_1^{-1} t_1^{-1} s_1 e_1 = 1$. Continuing in the same way we see that all corresponding elements are equal so $c = c'$.