PHY 5207: GEOMETRIC GROUP THEORY
LECTURE 6

HNN EXTENSIONS

Definition 4.11. Let G be a group, A a subgroup of G and $\theta : A \to G$ a monomorphism. The *HNN-extension* of G over A with respect to θ is the group

$$G \underset{A}{*} = \langle G * < t > | tat^{-1} = \theta(a), \forall a \in A \rangle = G * < t > / \langle\langle tat^{-1}\theta(a)^{-1}, a \in A \rangle\rangle$$

The letter t is called stable letter of the HNN-extension.

We remark that if $\langle S|R \rangle$ is a presentation of G then a presentation of $G \underset{A}{*}$ is given by

$$\langle S \cup \{t\} | R \cup \{tat^{-1} = \theta(a), \forall a \in A\} \rangle$$

Let A_1, A_2 be sets of right coset representatives of $A, \theta(A)$ in G so that $1 \in A_1, 1 \in A_2$. A *reduced word* of the HNN extension $G \underset{A}{*}$ is a word of the form

$$(g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, ..., t^{\epsilon_n}, g_n)$$

where $\epsilon_i = \pm 1$, $g_0 \in G$, $g_i \in A_1$ if $\epsilon_i = 1$, $g_i \in A_2$ if $\epsilon_i = -1$ and $g_i \neq 1$ if $\epsilon_{i+1} = -\epsilon_i$.

If $(g_0, t^{\epsilon_1}, ..., t^{\epsilon_n}, g_n)$ is a reduced word we associate to this the group element $g_0 t^{\epsilon_1} ... t^{\epsilon_n} g_n \in G \underset{A}{*}$.

Theorem 4.6. *(Normal forms) Each $g \in G \underset{A}{*}$ is represented by a unique reduced word.*

Proof. It is easy to see by successive reductions that any $g \in G \underset{A}{*}$ can be represented by some reduced word. We show now that this representation is unique. We use a similar argument as for amalgamated products. Let X be the set of all reduced words. We define an action of $G \underset{A}{*}$ on X. To do this it is enough to define actions of G and $< t >$ and show that the relations are satisfied. Let $g \in G$ and $(g_0, t^{\epsilon_1}, ..., t^{\epsilon_n}, g_n)$ a reduced word. We define

$$g \cdot (g_0, t^{\epsilon_1}, ..., t^{\epsilon_n}, g_n) = (gg_0, t^{\epsilon_1}, ..., t^{\epsilon_n}, g_n)$$

Clearly this defines an action of G on X. We define now the action of t.

$$t \cdot (g_0, t^{\epsilon_1}, ..., t^{\epsilon_n}, g_n) = \begin{cases} (\theta(a), t, g_0', t^{\epsilon_1}, ..., t^{\epsilon_n}, g_n) & \text{if } g_0 = ag_0',\ 1 \neq g_0' \in A_1 \\ (\theta(g_0), t, 1, t^{\epsilon_1}, ..., t^{\epsilon_n}, g_n) & \text{if } g_0 \in A, \epsilon_1 = 1 \\ (\theta(g_0)g_1, t^{\epsilon_2}, ..., t^{\epsilon_n}, g_n) & \text{if } g_0 \in A, \epsilon_1 = -1 \end{cases}$$

So t defines a 1-1 map $X \to X$. We show that this map is onto. If $(g_0, t^{\epsilon_1}, ..., t^{\epsilon_n}, g_n) \in X$ then

$$(g_0, t^{\epsilon_1}, ..., t^{\epsilon_n}, g_n) = \begin{cases} t \cdot (1, t^{-1}, g_0, t^{\epsilon_1}, ..., t^{\epsilon_n}, g_n) & \text{if } g_0 \notin \theta(A) \\ t \cdot (ag_1, t^{\epsilon_2}, ..., t^{\epsilon_n}, g_n) & \text{if } g_0 = \theta(a),\ a \in A, \epsilon_1 = 1 \\ t \cdot (a, t^{-1}, 1, t^{\epsilon_1}g_1, t^{\epsilon_2}, ..., t^{\epsilon_n}, g_n) & \text{if } g_0 = \theta(a),\ a \in A, \epsilon_1 = -1 \end{cases}$$

So t gives an element of $Symm(X)$. In other words we have defined homomorphisms $G \to Symm(X)$, $< t > \to Symm(X)$. It follows that there is an extension of these homomorphisms to $G* < t > \to Symm(X)$. We verify that tat^{-1} and $\theta(a)$ $(a \in A)$ act in the same way. So we have an action of $G \underset{A}{*}$ on X. If $g_0 t^{\epsilon_1} ... t^{\epsilon_n} g_n \in G \underset{A}{*}$ is an element corresponding to a reduced word then

$$g_0 t^{\epsilon_1} ... t^{\epsilon_n} g_n \cdot (1) = (g_0, t^{\epsilon_1}, ..., t^{\epsilon_n}, g_n)$$

So each element is represented by a unique reduced word. □

Corollary 4.6. *The group G embeds in $G \underset{A}{*}$.*

Corollary 4.7. *Let $G \underset{A}{*}$ be an HNN extension. Let $(g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, ..., t^{\epsilon_n}, g_n)$ be such that $g_i \in G$ for all i, $\epsilon_i = \pm 1$, $g_i \notin A$ if $\epsilon_i = 1$ and $\epsilon_{i+1} = -1$, $g_i \notin \theta(A)$ if $\epsilon_i = -1$ and $\epsilon_{i+1} = 1$, then $g_0 t^{\epsilon_1} g_1 ... t^{\epsilon_n} g_n \neq 1$ in $G \underset{A}{*}$.*

Proof. Starting from g_n we replace successively the g_i's by elements of the form hs_i where s_i lies in $A_1 \cup A_2$ (right coset representatives of $A, \theta(A)$) so that eventually we arrive at a reduced word representing $g_0 t^{\epsilon_1} g_1 ... t^{\epsilon_n} g_n$ which has length n, so $g_0 t^{\epsilon_1} g_1 ... t^{\epsilon_n} g_n \neq 1$. □

Definition 4.12. If a group G is an amalgam $G = A \underset{H}{*} B$ (with $A \neq H \neq B$) or an HNN-extension $G = A \underset{H}{*}$ then we say that G *splits* over H.

Example 4.2. (Higman, Neumann and Neumann) Any countable group embeds in a group with 2 generators.

Proof. Let $C = \{c_0 = e, c_1, c_2, ...\}$ be a countable group. We remark that the set of elements $S = \{a^n ba^{-n} : n \in \mathbb{N}\}$ forms a basis for free subgroup of the free group of rank 2, $F = F(a, b)$. Consider the group

$$H = F * C$$

The subgroups

$$A = \langle a^n ba^{-n} : n \in \mathbb{N}\rangle,\ B = \langle c_n b^n ab^{-n} : n \in \mathbb{N}\rangle$$

are both free of infinite rank by the normal form theorem for free products (theorem 4.3). Let $\phi : A \to B$ be the isomorphism given by $\phi(a^n ba^{-n}) = c_n b^n ab^{-n}$. Consider the HNN extension

$$G = H \underset{A}{*} = \langle H* < t > |ta^n ba^{-n} t^{-1} = c_n b^n ab^{-n},\ \forall n \in \mathbb{N}\rangle$$

Clearly C embeds in G (normal form theorem for HNN extensions). Morover

$$ta^n ba^{-n} t^{-1} = c_n b^n ab^{-n} \implies c_n = ta^n ba^{-n} t^{-1} b^n a^{-1} b^{-n}$$

so G is generated by t, a, b, and in fact since $tbt^{-1} = a$, G is generated by a, t. □