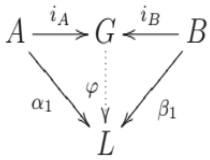
**PHY 5207: GEOMETRIC GROUP THEORY**
**LECTURE 5**

# Amalgams

The construction of amalgams allows us to 'combine' some given groups and construct new groups. Let $A, B$ be two groups which have two isomorphic subgroups, that is there are embeddings $\alpha : H \to A,\ \beta : H \to B$. Intuitively the amalgam of $A, B$ over $H$ is a group that contains copies of $A, B$ which intersect along $H$ and no other relations are imposed. To simplify notation we pose $\alpha(h) = h,\ \beta(h) = \bar{h}$ for all $h \in H$.

One way to define amalgams is via their universal property:

**Definition 4.9.** We say that a group $G$ is the *amalgamated product* of $A, B$ over $H$ and we write $G = A \underset{H}{*} B$ if there are homomorphisms $i_A : A \to G,\ i_B : B \to G$ which agree on $H$ such that for every group $L$ and homomorphisms $\alpha_1 : A \to L,\ \beta_1 : B \to L$ which satisfy $\alpha_1(h) = \beta_1(\bar{h}),\ \forall h \in H$, there is a unique homomorphism $\varphi : G \to L$ such that $\alpha_1 = \varphi \circ i_A$ and $\beta_1 = \varphi \circ i_B$.




The amalgam of $A, B$ over $H$ depends of course on the maps $\alpha, \beta$, it is however customary to suppress this on the notation. We note that it is not clear by the definition whether $i_A, i_B$ are injective.

*Remark* 4.5. Assuming that an amalgam of $A, B$ over $H$ exists it is easy to see that this amalgam is unique using the universal property.

Indeed let $G_1, G_2$ be two such amalgams and let $i_A, i_B, j_A, j_B$ be the inclusions of $A, B$ in $G_1, G_2$ respectively. The homomorphisms $j_A, j_B$ induce a homomorphism $j : G_1 \to G_2$ such that $j \circ i_A = j_A,\ j \circ i_B = j_B$. Similarly $i_A, i_B$ induce a homomorphism $i : G_2 \to G_1$. The compositions of these maps induce homomorphisms $G_1 \to G_1,\ G_2 \to G_2$ which are both equal to the identity since they are induced by $i_A, i_B$ and $j_A, j_B$ respectively. So $G_1 \cong G_2$.

We show now that the amalgam of $A, B$ over $H$ exists:

Let $\langle S_1 | R_1 \rangle,\ \langle S_2 | R_2 \rangle$ be presentations of $A, B$ respectively. Without loss of generality we assume that $S_1 \cap S_2 = \emptyset$. Then the amalgam of $A, B$ over $H$ is given by

$$A \underset{H}{*} B = \langle S_1 \cup S_2 | R_1 \cup R_2 \cup \{h = \bar{h} : h \in H\} \rangle$$

Indeed it is easy to see that this group satisfies the universal property of the definition.

When $H = \{1\}$ then the amalgam does not depend on the maps $\alpha, \beta$ and it is called *free product* of $A, B$; we denote this by $A * B$. We remark that $F_2 = \mathbb{Z} * \mathbb{Z}$. We would like to describe the elements of $A \underset{H}{*} B$ by 'words'. To simplify notation we identify $H$ with its image in $A, B$. If $a \in A$ (or $b \in B$) we will denote the corresponding element of $G$ by $a$ ($b$) rather than $i_A(a)$ ($i_B(b)$). It is important to distinguish whether we see $a$ as an element of $A$ or of $G$ since, a priori, it is possible that $a_1 = a_2$ in $G$ while $a_1 \neq a_2$ in $A$ (and similarly for $B$).

Let $A_1, B_1$ be sets of right coset representatives of $H$ in $A, B$ respectively, such that $1 \in A_1, 1 \in B_1$. So we have the 1-1 and onto maps:

$$H \times A_1 \to A, (h, a) \mapsto ha, \; H \times B_1 \to B, (h, b) \mapsto hb$$

A *reduced word* of the amalgam $A \underset{H}{*} B$ is a word of the form $(h, s_1, ..., s_n)$ where $h \in H$, $s_i \in A_1 \cup B_1$, $s_i \neq 1$ for every $i$ and the $s_i$'s alternate from $A_1$ to $B_1$. That is for all $i$, $s_i \in A_1 \implies s_{i+1} \in B_1$, $s_i \in B_1 \implies s_{i+1} \in A_1$. If $(h, s_1, ..., s_n)$ is a reduced word we associate to this the group element $hs_1...s_n \in A \underset{H}{*} B$. We say that the *length* of the reduced word $(h, s_1, ..., s_n)$ is $n$.

**Theorem 4.3.** *(Normal forms) Each $g \in G = A \underset{H}{*} B$ is represented by a unique reduced word.*

*Proof.* Any element $g \in G$ can be written as a product of the form

$$g = a_1 b_1 ... a_n b_n, \; a_i \in A, b_i \in B$$

By successive reductions we arrive at a reduced word, so we can represent $g$ by a reduced word. We show now that this word is unique.

Let $X$ be the set of all reduced words. We define an action of $G$ on $X$. We recall that an action is a homomorphism $G \to Symm(X)$. By the universal property of the amalgam it is enough to define homomorphisms (actions) $A \to Symm(X)$, $B \to Symm(X)$ which agree on $H$. We define the action of $A$. If $a \in H$ and $(h, s_1, ..., s_n)$ is a reduced word we define

$$a \cdot (h, s_1, ..., s_n) = (ah, s_1, ..., s_n)$$

If $a \in A \setminus H$ and $(h, s_1, ..., s_n)$ a reduced word then there are two cases.
1st case: $s_1 \in B$. Then $ah = h_1 s$ for some $h_1 \in H$, $s \in A_1$ and we define

$$a \cdot (h, s_1, ..., s_n) = (h_1, s, s_1, ..., s_n)$$

2nd case: $s_1 \in A$. Then $ahs_1 = h_1 s$ for some $h_1 \in H$, $s \in A_1$. If $s \neq 1$ we define

$$a \cdot (h, s_1, ..., s_n) = (h_1, s, s_2, ..., s_n)$$

while if $s = 1$ we define

$$a \cdot (h, s_1, ..., s_n) = (h_1, s_2, ..., s_n)$$

One sees easily that if $a_1, a_2 \in A$ then

$$(a_1 a_2) \cdot (h, s_1, ..., s_n) = a_1 \cdot (a_2 \cdot (h, s_1, ..., s_n))$$

so we have indeed an action. We define the action of $B$ similarly. So we have an action of $G$ on $X$. Now if $g = hs_1 ... s_n$ where $(h, s_1, ..., s_n)$ is a reduced word then

$$g \cdot (1) = (h, s_1, ..., s_n)$$

It follows that the reduced word representing $g$ is unique. □

**Corollary 4.2.** *The homomorphisms* $i_A : A \to A \underset{H}{*} B$, $i_B : B \to A \underset{H}{*} B$ *are injective. So we can see* $A, B$ *as subgroups of* $A \underset{H}{*} B$.

From now on we may identify elements of $A \underset{H}{*} B$ with reduced words.

**Corollary 4.3.** *Let* $A \underset{H}{*} B$ *be an amalgamated product. If* $(g_1, ..., g_n)$ *is such that* $g_i \in A \cup B$ , $g_i \notin H$ *for any* $i > 1$ *and the* $g_i$*'s alternate between* $A$ *and* $B$ *then* $g_1 g_2 ... g_n \neq 1$ *in* $A \underset{H}{*} B$.

*Proof.* Starting from $g_n$ we replace successively the $g_i$'s by elements of the form $hs_i$ where $s_i$ lies in $A_1 \cup B_1 \setminus 1$ (right coset representatives of $H$). Eventually we arrive at a reduced word representing $g_1 g_2 ... g_n$ which has length $n$ if $g_1 \notin H$, and $n - 1$ if $g_1 \in H$. It follows that $g_1 g_2 ... g_n \neq 1$. □

**Exercise 4.1.** Show that if $A \neq H \neq B$ then the center of $A \underset{H}{*} B$ is contained in $H$.

If $hs_1 ... s_n$ is a reduced word (element) in $A \underset{H}{*} B$ then we say that $n$ is the length of this word. We say that a reduced element $hs_1 ... s_n$ $(n > 1)$ is *cyclically reduced* if $s_1 s_n$ is reduced.

**Proposition 4.1.** *1. Every element of* $A \underset{H}{*} B$ *is conjugate either to a cyclically reduced element or to an element of* $A$ *or* $B$.

*2. Every cyclically reduced element has infinite order.*

*Proof.* 1. If $g = hs_1...s_n$ is not cyclically reduced then $g$ is conjugate to an element of length $n-1$. We repeat till we arrive either at a reduced word or an element of $A$ or $B$.

2. If $g$ is cyclically reduced of length $n$ then $g^k$ has length $kn$ so $g^k \neq 1$. □

**Exercise 4.2.** If $K$ is a finite subgroup of $A \underset{H}{*} B$ then $K$ is contained in a conjugate of either $A$ or $B$.

**Example 4.1.** (Higman) Let

$$A = \langle a, s | sas^{-1} = a^2 \rangle$$

$$B = \langle b, t | tbt^{-1} = b^2 \rangle$$

Then $< a > \cong < b > \cong \mathbb{Z}$ so we may form the amalgam

$$G = A \underset{<a>=<b>}{*} B = \langle a, s, t | sas^{-1} = a^2, \, tat^{-1} = a^2 \rangle$$

The group $G$ is not Hopf.

*Proof.* We define $\varphi : G \to G$ by

$$\varphi(a) = a^2, \, \varphi(s) = s, \, \varphi(t) = t$$

It is easy to see that the relations are satisfied so $\varphi$ is a homomorphism. Moreover $\varphi(t^{-1}at) = t^{-1}a^2t = a$ so $\varphi$ is onto. On the other hand $\varphi(s^{-1}ast^{-1}a^{-1}t) = s^{-1}a^2st^{-1}a^{-2}t = aa^{-1} = 1$. As $s^{-1}as \in A - < a >$, $t^{-1}a^{-1}t \in B - < b >$ (check this! see example 3.1) the element $(s^{-1}as)(t^{-1}a^{-1}t)$ has length 2 in the amalgam $A \underset{<a>=<b>}{*} B$ so $ker\, \varphi \neq 1$.

□

# Actions of amalgams on Trees

**Definition 4.10.** Let $G$ be a group acting without inversions on a tree $T$. A subtree $S \subset T$ is called a *fundamental domain* of the action if the standard projection $p : S \to T/G$ is an isomorphism.

**Theorem 4.4.** *Let $G = A \underset{H}{*} B$ be an amalgamated product. Then $G$ acts on a tree $T$ with fundamental domain an edge $e = [P, Q]$ so that $stab(P) = A$, $stab(Q) = B$, $stab(e) = H$.*

*Proof.* We define the vertices of $T$ to be

$$V(T) = G/A \sqcup G/B = \{gA : g \in G\} \sqcup \{gB : g \in G\}$$

and the edges

$$E(T) = G/H \sqcup \overline{G/H}$$

We define $o(gH) = gA$, $t(gH) = gB$. The action of $G$ is the obvious one: If $g' \in G$ then

$$g' \cdot gA = (g'g)A, \quad g' \cdot gB = (g'g)B, \quad g' \cdot gH = (g'g)H$$

Clearly $G$ acts transitively on the set of geometric edges of $T$ and there are two orbits of vertices. $T$ is connected since if $g = hs_1...s_n$, (reduced word of length $n$) then there is an edge joining $gA$ to $hs_1...s_{n-1}B$ if $s_n \in A$. Otherwise there is an edge joining $gB$ to $hs_1...s_{n-1}A$. Since $gA$, $gB$ are joined by an edge we see by induction on the length of $g$ that every vertex $gA$ or $gB$ can be joined by a path to $1 \cdot A$, so $T$ is connected.

We note that if $p$ a path starting and ending at $1 \cdot A$ then necessarily the length of $p$ is even. Suppose now that $p$ is a reduced path of length $2n$ starting at $1 \cdot A$. We claim that the vertices of $p$ are of the form

$$1 \cdot A, a_1B, a_1b_1A, ..., a_1b_1...a_nb_nA$$

where $a_i \in A - H$ for $i > 1$ and $b_i \in B - H$ for all $i$. Indeed this is easily proven inductively as if e.g. $a_1b_1...a_kb_kA, gB$ are successive vertices then $gb = a_1b_1...a_kb_ka$ for some $a \in A, b \in B$. However $gbB = gB$ so we may denote the vertex $gB$ by $a_1b_1...a_kb_kaB$ (in other words $a_{k+1} = a$). Note also that if $a \in H$ then $gB = a_1b_1...a_kB$ so the path is not reduced. It follows that the length of $a_1b_1...a_nb_n$ is at least $2n - 1$ so $1A \neq a_1b_1...a_nb_nA$, ie there are no reduced paths starting and ending at $A$. Similarly there are no reduced paths starting and ending at $B$. As every vertex of $T$ lies either in the orbit of $A$ or of $B$ we conclude that $T$ has no circuits.

Therefore $T$ is a tree. □

**Corollary 4.4.** *Let $F$ be a subgroup of $A \underset{H}{*} B$ which intersects trivially any conjugate of $A$ or $B$. Then $F$ is free.*

*Proof.* Let $T$ be the tree constructed in the theorem 4.4. The stabilizers of vertices of $T$ are conjugates of $A, B$. Since $F$ intersects trivially the conjugates of $A, B$, $F$ acts freely on $T$. By theorem 4.2 $F$ is a free group. □

**Proposition 4.2.** *Let $G = A*B$. Then the kernel of the natural map $\varphi : A*B \to A \times B$ is free.*

*Proof.* If $R = ker\,\varphi$ then $R$ intersects trivially all conjugates of $A, B$ since these map isomorphically to their image. By corollary 4.4 $R$ is free. □

**Corollary 4.5.** *If $A, B$ are finite groups then $A * B$ has a finite index subgroup which is free.*

Theorem 4.4 has a converse:

**Theorem 4.5.** *Assume that $G$ acts on a tree $T$ with fundamental domain an edge $e = [P, Q]$. If $stab(P) = A$, $stab(Q) = B$, $stab(e) = H$ then $G = A \underset{H}{*} B$.*

*Proof.* The inclusions $A \to G$, $B \to G$ induce a homomorphism

$$\varphi : A \underset{H}{*} B \to G$$

We consider the subgroup $G' = <A, B>$. We remark that $G'e$ is connected. If for some $g_1 \in G$, $g_2 \in G'$ we have that $g_1 P = g_2 P$ then $g_2^{-1} g_1 \in A$ so $g_1 \in G'$. The same holds if $g_1 Q = g_2 Q$. So $(G - G')e \cap G'e = \emptyset$. On the other hand $T = Ge = (G - G')e \cup G'(e)$ and $T$ is connected. It follows that $G - G' = \emptyset$ and $G = G'$. Therefore $\varphi$ is onto. We show now that $\varphi$ is 1-1. Let $g = hs_1 ... s_n$ (reduced word in $A \underset{H}{*} B$) be an element of $ker\,\varphi$. Clearly $n > 1$.

We distinguish now two cases. If $s_n \in A$ then we see by induction on $n$ that $d(gQ, Q) = n$ if $n$ is even and $d(gQ, Q) = n + 1$ if $n$ is odd. Similarly if $s_n \in B$ we see inductively that $d(gP, P) = n$ if $n$ is even and $d(gP, P) = n + 1$ if $n$ is odd. It follows that $g \neq 1$ in $G$ so $\varphi$ is 1-1.