GROUP ACTIONS ON TREES

Group actions on sets

We recall the definition of a group action on a set:

Definition 4.1. Let G be a group and let X be a set. An *action* of G on X is a map

$$\rho : G \times X \to X$$

so that the following hold:

1. $\rho(1, x) = x$ for all $x \in X$.
2. $\rho(g_1 g_2, x) = \rho(g_1, \rho(g_2, x))$ for all $g_1, g_2 \in G, x \in X$.

We often write simply $g(x)$ or gx instead of $\rho(g, x)$. Note that by property 2, $g^{-1}(gx) = x$. It follows that the map

$$x \mapsto gx$$

is 1-1 and onto map from X to X.

In fact we have an equivalent definition of a group action as a homomorphism:

$$\varphi : G \to Symm(X)$$

Indeed if $\rho : G \times X \to X$ is an action define $\varphi : G \to Symm(X)$ by $\varphi(g)(x) = gx$. Property 2 of the definition implies that $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$. Conversely given a homomorphism φ we define $\rho : G \times X \to X$ by $\rho(g, x) = \varphi(g)(x)$.

We often denote an action by $G \curvearrowright X$.

Graphs

Definition 4.2. A *graph* Γ consists of a set of vertices $V = V(\Gamma)$, a set of edges $E = E(\Gamma)$, a map:

$$E \to V \times V, \ \ e \mapsto (o(e), t(e))$$

and a map $E \to E$, $e \mapsto \bar{e}$ such that the following hold: for any $e \in E$, $\bar{\bar{e}} = e$, $\bar{e} \neq e$ and $o(\bar{e}) = t(e)$, $t(\bar{e}) = o(e)$.

The pair of edges $\{e, \bar{e}\}$ is called *geometric edge*. Often when we define graphs we just give the vertices and the geometric edges of the graph. A choice of edges $E^+ \subset E(\Gamma)$ such that for any $e \in E(\Gamma)$ either $e \in E^+$ or $\bar{e} \in E^+$ is called an *orientation* of Γ.

A morphism between two graphs is a map that preserves the graph structure. More formally we have:

Definition 4.3. A *morphism* f from a graph $\Gamma = (V(\Gamma), E(\Gamma))$ to graph $\Delta = (V(\Delta), E(\Delta))$ is given by maps $f_V : V(\Gamma) \to V(\Delta)$, $f_E : E(\Gamma) \to E(\Delta)$ such that $o(f(e)) = f(o(e))$, $t(f(e)) = f(t(e))$, $f(\bar{e}) = \overline{f(e)}$. An automorphism of Γ is a morphism $\Gamma \to \Gamma$ that is 1-1 and onto on the sets of edges and vertices. We denote by $Aut(\Gamma)$ the group of automorphisms of Γ.

Definition 4.4. Let $G =< S >$ be a group generated by S. We define the *Cayley graph* of G, $\Gamma = \Gamma(S, G)$, to be the graph with vertices $V(\Gamma) = \{g : g \in G\}$ and oriented edges $E^+(\Gamma) = \{(g, gs) : g \in G, \, s \in S\}$. We define $o(g, gs) = g$, $t(g, gs) = gs$.

More generally if $S \subset G$, where S is not necessarily a generating set we define the graph $\Gamma(S, G)$ as before to be the graph with vertices $\{g : g \in G\}$ and oriented edges $\{(g, gs) : g \in G, \, s \in S\}$.

Definition 4.5. A *path* in a graph Γ is a sequence of edges $p = (e_1, ..., e_n)$ such that $o(e_i) = t(e_{i-1})$ for all $i > 1$. The vertices $x = o(e_1)$, $y = t(e_n)$ are the *origin* and the *end point* of the path respectively. We often say that p joins x, y. We define similarly infinite paths. We say that a path is *reduced* if $e_i \neq \bar{e}_{i-1}$ for all $i > 1$. We say that a path $(e_1, ..., e_n)$ is a *circuit* if it is reduced, the vertices $t(e_i)$ $(i = 1, ..., n)$ are all distinct and $t(e_n) = o(e_1)$. We say that a graph is *connected* if any two vertices can be joined by a path. A *tree* is a connected graph with no circuits.

Remark 4.1. The Cayley graph of G is a connected graph. Conversely if if $\Gamma(S, G)$ is connected for some $S \subset G$, then S is a generating set of G.

Remark 4.2. A graph Γ is a tree if and only if for any two vertices of Γ there is a unique reduced path joining them (exercise).

One may realize graphs as 1-dimensional CW-complexes: we start with a set of points (vertices) and glue edges to them; so if $e = [0, 1]$ is an edge we glue 0 to $o(e)$ and 1 to $t(e)$. The edges $e, \bar{e}$ correspond geometrically to the same edge, and $e, \bar{e}$ are thought of as the two possible orientations of this edge. We can equip a connected graph with a metric by identifying each edge with an interval of length 1 and defining the distance of two points to be the length of the shortest path joining them.

Definition 4.6. An *action* of a group G on a graph Γ is a homomorphism $\rho : G \to Aut(\Gamma)$.

If $\rho : G \to Aut(\Gamma)$ is an action, $g \in G$ and $v \in V(\Gamma)$ then $\rho(g)(v) \in V(\Gamma)$. Usually we simplify the notation and we write gv rather than $\rho(g)(v)$. If G acts on Γ we write also $G \curvearrowright \Gamma$. If there is some $v \in V(\Gamma)$ such that $gv = v$ for all $g \in G$ then we say that G fixes a vertex of Γ.

Remark 4.3. A group $G =< S >$ acts on its Cayley graph $\Gamma(S, G)$ as follows: If $g \in G$ and (v, vs) an edge of $\Gamma(S, G)$ we define $g \cdot (v, vs) = (gv, gvs)$. We remark that this action is transitive on the set of vertices of $\Gamma(S, G)$.

4.3 Actions of free groups on Trees

Theorem 4.1. *Let S be a subset of a group G, and let $X = \Gamma(S, G)$. The following are equivalent:*

i) X is a tree.

ii) G is free with basis S.

Proof. ii) $\Longrightarrow$ i).

Assume that G is free with basis S. Every element of G can be represented by a reduced word on S, $s_1 ... s_n$. There is a path from 1 to $s_1 ... s_n$:

$$p = ((1, s_1), (s_1, s_1 s_2), ..., (s_1 s_2 ... s_{n-1}, s_1 s_2 ... s_{n-1} s_n))$$

so X is connected. In general a reduced path starting at 1 corresponds to a reduced word on S, w. Since reduced words represent non trivial elements in G we have that $w \neq 1$ in G, so there are no circuits staring at 1. However since the action of G is transitive on vertices we deduce that X has no circuits, hence it is a tree.

i) $\Longrightarrow$ ii)

Since X is connected there is a reduced path from 1 to any $g \in G$. Therefore any $g \in G$ can be written as a word on S. It follows that S generates G. Let $\varphi : F(S) \to G$

be the onto homomorphism defined by $\varphi(s) = s$ for all $s \in S$. Then if $s_1 ... s_n \in ker\,\varphi$ ($s_1 ... s_n$ reduced word) we have that the path

$$p = ((1, s_1), (s_1, s_1 s_2), ..., (s_1 s_2 ... s_{n-1}, s_1 s_2 ... s_{n-1} s_n))$$

is a reduced path in X from 1 to 1, which is impossible. We conclude that φ is 1-1, so $G \cong F(S)$.

□

Definition 4.7. Let G be a group acting on a graph X. We say that G acts on X *without inversions* if for every $g \in G$, $e \in E(X)$ we have that $ge \neq \bar{e}$. We say that G acts *freely* on X if G acts on X without inversions and for any $1 \neq g \in G$, $v \in V(X)$, $gv \neq v$.

Remark 4.4. A group $G = < S >$ acts without inversions on the Cayley graph $\Gamma(S, G)$.

Note that if G acts on a graph Γ then it acts without inversions on the barycentric subdivision of Γ (i.e. the graph obtained by subdividing each edge of Γ in two edges).

Definition 4.8. Let G be a group acting without inversions on a graph X. We define the *quotient graph* of the action X/G as follows: If $v \in V(X)$, $e \in E(X)$ we set

$$[v] = \{gv : g \in G\}, \ \ [e] = \{ge : g \in G\}$$

The vertices and edges of the quotient graph are given by

$$V(X/G) = \{[v] : v \in V(X)\}, \ E(X/G) = \{[e] : e \in E(X)\}$$

and $o([e]) = [o(e)]$, $t([e]) = [t(e)]$, $\overline{[e]} = [\bar{e}]$.

We remark that since the action is without inversions $[\bar{e}] \neq [e]$. There is an obvious graph morphism

$$p : X \to X/G, \text{ given by } p(v) = [v], \ p(e) = [e], \ v \in V(X), e \in E(X)$$

Theorem 4.2. *If a group G acts freely on a tree T then G is free.*

Proof.

Lemma 4.1. *There is a tree $X \subset T$ such that X contains exactly one vertex from each orbit of the action.*

Proof. Let X be a maximal subtree of T such that X contains at most one vertex from each orbit. Clearly such a tree exists by Zorn's lemma. Suppose that X does not intersect all orbits of vertices. Let v be a vertex of minimal distance from X such that X does not meet its orbit. If $d(v, X) = 1$ then we can add v to X contradicting its maximality. Otherwise if p is a reduced path from v to X and v' is the first vertex of p then $gv' \in X$ for some $g \in G$. But then $d(gv, X) = 1$ so we can add gv to X, a contradiction. We conclude that X contains exactly one vertex from each orbit. □

Let X be as in the lemma. We choose an orientation of the edges of T, $E^+ \subset E(T)$ such that E^+ is invariant under the action (that is $e \in E^+ \Rightarrow ge \in E^+$, for all $g \in G$). This is possible since the action is without inversions.

Consider the set

$$S = \{g \in G : \text{ there is an edge } e \in E^+ \text{ with } o(e) \in X,\ t(e) \in g(X)\}$$

We will show that G is a free group with basis S.

Clearly if $g_1 \neq g_2$ then $g_1X \cap g_2X = \emptyset$. Let T' be the tree that we obtain from T by contracting each translate gX to a point. Clearly G acts on T'. We will show that $T' \simeq \Gamma(S, G)$. We remark that $V(T') = \{gX : g \in G\}$, $E(T') = \{e \in T,\ e \notin GX\}$. The orientation of T induces an orientation of the edges of T' which we denote still by E^+. We define now $\varphi : T' \to \Gamma(S, G)$ as follows: $\varphi(gX) = g$. If $e \in E^+$ is an edge joining g_1X to g_2X then $s = g_1^{-1}g_2 \in S$ since $g_1^{-1}e$ joins X to $g_1^{-1}g_2X$. So we define $\varphi(e) = (g_1, g_1s) = (g_1, g_2)$. It is clear that φ is 1-1 and onto on the set of vertices $V(T')$. It is also onto on oriented edges: if (g, gs) is an oriented edge of $\Gamma(S, G)$ then there is an oriented edge $e \in T'$ joining X to sX and $\varphi(ge) = (g, gs)$. We note that if

$$\varphi(e_1) = \varphi(e_2) = (g, gs)$$

then e_1, e_2 are both oriented edges joining gX to gsX. But T' is a tree so $e_1 = e_2$ and φ is 1-1.

It follows that $\Gamma(S, G)$ is a tree, hence G is free (theorem 4.1).

□

Corollary 4.1. *Subgroups of free groups are free.*

Proof. Let $F(S)$ be a free group with basis S. Then $F(S)$ acts freely on its Cayley graph $\Gamma(S, G)$ which is a tree. So any subgroup H of $F(S)$ acts freely on $\Gamma(S, G)$ hence by the previous theorem H is free. □