

## GROUP PRESENTATIONS (DEHN'S PROBLEMS)

**Definition 3.1.** A *presentation*  $P$  is a pair  $P = \langle S|R \rangle$  where  $S$  is a set and  $R$  is a set of words in  $S$ . The group defined by  $P$  is the quotient group

$$G = F(S)/\langle\langle R \rangle\rangle$$

where  $\langle\langle R \rangle\rangle$  is the smallest normal subgroup of the free group  $F(S)$  that contains  $R$ . By abuse of notation we write often  $G = \langle S|R \rangle$ .

*Remark 3.1.* From corollary 2.1 it follows that every group has a presentation.

A group  $G$  is called *finitely generated* if there are finitely many elements of  $G$ ,  $g_1, \dots, g_n$  such that any element of  $g$  can be written as a product of  $g_i^{\pm 1}$ ,  $i = 1, \dots, n$ . Clearly if  $G$  is finitely generated then  $G$  has a presentation  $\langle S|R \rangle$  with  $S$  finite. We say that a group  $G = \langle S|R \rangle$  is *finitely related* if  $R$  is finite. If both  $S$  and  $R$  are finite we say that  $G$  is *finitely presented*.  $S$  is the set of generators and  $R$  is the set of relators of the presentation. Sometimes we write relators as equations, so instead of writing  $r$  we write  $r = 1$  or even  $r_1 = r_2$ , which is of course equivalent to  $r_1 r_2^{-1} = 1$ .

**Examples.** 1. A presentation of  $\mathbb{Z}$  is given by  $\langle a \rangle$ .

2. A presentation of  $\mathbb{Z}_n$  is given by  $\langle a|a^n \rangle$ .

3. A presentation of the free group  $F(S)$  is given by  $\langle S \rangle$ .

4. A presentation of  $\mathbb{Z} \oplus \mathbb{Z}$  is given by  $\langle a, b|aba^{-1}b^{-1} \rangle$ .

Indeed if  $\varphi : F(a, b) \rightarrow \mathbb{Z} \oplus \mathbb{Z}$  is the homomorphism defined by  $\varphi(a) = (1, 0)$ ,  $\varphi(b) = (0, 1)$  then clearly  $aba^{-1}b^{-1} \in \ker \varphi$ . We set  $N = \langle\langle aba^{-1}b^{-1} \rangle\rangle$ . Since  $aba^{-1}b^{-1} \in \ker \varphi$ ,  $N \subset \ker \varphi$ . We remark that in  $F(a, b)/N$  we have that  $ab = ba$ . If

$$w = a^{k_1} b^{m_1} \dots a^{k_n} b^{m_n} \in \ker \varphi$$

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then  $\sum k_i = \sum m_i = 0$ . Therefore  $w = 1$  in  $F(a, b)/N$  since  $ab = ba$  in this quotient group. It follows that  $\ker \varphi \subset N$  and  $\langle a, b | aba^{-1}b^{-1} \rangle$  is a presentation of  $\mathbb{Z} \oplus \mathbb{Z}$ .

5. If  $G$  is a finite group,  $G = \{g_1, \dots, g_n\}$  then a presentation of  $G$  is:  $\langle G | R \rangle$  where  $R$  is the set of the  $n^2$  equations of the form  $g_i g_j = g_k$  given by the multiplication table of  $G$ .

6. The presentation  $\langle a, b | a^{-1}ba = b^2, b^{-1}ab = a^2 \rangle$  is a presentation of the trivial group. Indeed

$$a^{-1}ba = b^2 \implies (b^{-1}a^{-1}b)a = b \implies a^{-1} = b \implies a = 1 = b$$

*Remark 3.2.* Let  $G = \langle S | R \rangle$ . Then a word  $w$  on  $S$  represents the identity in  $G$  if and only if  $w$  lies in the normal closure of  $R$  in  $F(S)$ . Equivalently if  $w$  can be written in  $F(S)$  as a product of conjugates of elements of  $R$  :

$$w = \prod_{i=1}^n x_i r_i^{\pm 1} x_i^{-1}, \quad r_i \in R, \quad x_i \in F(S)$$

We note that if  $w$  represents the identity in  $G$  we could prove that it is the case by listing all expressions of this form. Eventually we will find one such expression that is equal to  $w$  in  $S$ . Of course this presupposes that we know that  $w = 1$  in  $G$ , otherwise this process will never terminate.

**Proposition 3.1.** *Let  $G = \langle S | R \rangle$  and let  $H$  be a group. If  $\varphi : S \rightarrow H$  is a function then  $\varphi$  can be extended to a homomorphism  $\bar{\varphi} : G \rightarrow H$  if and only if  $\varphi(r) = 1$  for every  $r \in R$ , where if  $r$  is the word  $a_1^{\pm 1} \dots a_n^{\pm 1}$ , we define  $\varphi(r) = \varphi(a_1)^{\pm 1} \dots \varphi(a_n)^{\pm 1}$ .*

*Proof.* It is obvious that  $\varphi(r) = 1$  for every  $r \in R$  is a necessary condition for  $\varphi$  to extend to a homomorphism.

Clearly  $\varphi$  extends to  $\varphi : F(S) \rightarrow H$ . Assume now that  $\varphi(r) = 1$  for every  $r \in R$ . If  $N = \langle\langle R \rangle\rangle$  then clearly  $N \subset \ker \varphi$ . So the map  $\bar{\varphi}(aN) = \varphi(a)$  is a well defined homomorphism from  $G = F(S)/N$  to  $H$  that extends  $\varphi$ .

□

One can use this proposition to show that a group given by a presentation is non trivial by finding a non trivial homomorphism to another group.

Before the next example we recall the definition of the semidirect product:

Let  $A, B$  be groups and let  $\varphi : B \rightarrow \text{Aut}(A)$  be a homomorphism. Then we define the semidirect product of  $A$  and  $B$  to be the group  $G = A \rtimes_{\varphi} B$  with elements the elements of the Cartesian product  $A \times B$  and operation defined by

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 \varphi(b_1)(a_2), b_1 b_2) .$$

**Example 3.1.** If  $G = \langle a, t | tat^{-1} = a^2 \rangle$  then  $\langle t \rangle \cong \langle a \rangle \cong \mathbb{Z}$ .

*Proof.* Consider the subgroup of  $\mathbb{Q}$ :

$$\mathbb{Z}\left[\frac{1}{2}\right] = \left\{ \frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\}$$

We define an isomorphism  $\varphi : \mathbb{Z}\left[\frac{1}{2}\right] \rightarrow \mathbb{Z}\left[\frac{1}{2}\right]$ , by  $\varphi(x) = 2x$ . We form now the semidirect product  $\mathbb{Z}\left[\frac{1}{2}\right] \rtimes \mathbb{Z}$  where  $\mathbb{Z}$  acts on  $\mathbb{Z}\left[\frac{1}{2}\right]$  via  $\varphi$ . The elements of this semidirect product can be written as pairs  $\left(\frac{m}{2^n}, k\right)$ . We define now

$$\psi : G \rightarrow \mathbb{Z}\left[\frac{1}{2}\right] \rtimes \mathbb{Z}, \quad \text{by } \psi(a) = (1, 0), \quad \psi(t) = (0, 1).$$

Since

$$\psi(tat^{-1}) = \psi(a^2) = (2, 0),$$

$\psi$  is a homomorphism. Since  $a, t$  map to infinite order elements we have that

$$\langle t \rangle \cong \langle a \rangle \cong \mathbb{Z}.$$

□

**Exercise 3.1.** Show that a finite index subgroup of a finitely generated group is finitely generated.

## 3.1 Dehn's problems

Dehn posed in 1911 the following fundamental algorithmic problems:

1. **Word problem.** Given a finite presentation  $G = \langle S | R \rangle$  is there an algorithm to decide whether any word  $w$  on  $S$  is equal to 1 in  $G$ ?
2. **Conjugacy problem.** Given a finite presentation  $G = \langle S | R \rangle$  is there an algorithm to decide whether any words  $w, v$  on  $S$  represent conjugate elements of  $G$ ?
3. **Isomorphism problem.** Is there an algorithm to decide whether any two groups  $G_1, G_2$  given by finite presentations  $G_1 = \langle S_1 | R_1 \rangle, G_2 = \langle S_2 | R_2 \rangle$  are isomorphic?

All these problems were shown to be unsolvable in general by Novikov (1955) and independently by Boone (1959). Adyan (1957) and Rabin (1958) showed that there is no algorithm to decide whether a given presentation is a presentation of the trivial group.

## 3.2 Tietze transformations

Different presentations of the same group are related via Tietze transformations. There are two types of Tietze transformations:

(T1) If  $\langle S|R \rangle$  is a presentation and  $r \in \langle\langle R \rangle\rangle \subset F(S)$  then T1 is the replacement of  $\langle S|R \rangle$  by  $\langle S|R \cup \{r\} \rangle$ . Clearly these two presentations define isomorphic groups, an isomorphism  $\phi$  is defined on the generators by  $\phi(s) = s$  for all  $s \in S$ .

We denote also by T1 the inverse transformation.

(T2) If  $\langle S|R \rangle$  is a presentation,  $a \notin S$  and  $w \in F(S)$  then T2 is the replacement of  $\langle S|R \rangle$  by  $\langle S \cup \{a\} | R \cup \{a^{-1}w\} \rangle$ . Clearly these two presentations define isomorphic groups. A homomorphism  $\phi$  is defined on the generators by  $\phi(s) = s$  for all  $s \in S$ . One verifies easily that the inverse of  $\phi$  is given by  $\psi(s) = s$  for all  $s \in S$  and  $\psi(a) = w$ .

We denote also by T2 the inverse transformation.

**Theorem 3.1.** *Two finite presentations  $\langle S_1|R_1 \rangle, \langle S_2|R_2 \rangle$  define isomorphic groups if and only if they are related by a finite sequence of Tietze transformations.*

*Proof.* It is clear that if two presentations are related by a finite sequence of Tietze transformations they define isomorphic groups. Conversely suppose that  $G_1 = \langle S_1|R_1 \rangle \cong \langle S_2|R_2 \rangle = G_2$ . We may assume that  $S_1 \cap S_2 = \emptyset$ . Indeed if this is not the case using moves T1, T2 we can replace  $S_1$  by a set of letters with the same cardinality, disjoint from  $S_2$ . We consider now isomorphisms

$$\varphi : G_1 \rightarrow G_2, \quad \psi = \varphi^{-1} : G_2 \rightarrow G_1$$

For each  $s \in S_1, t \in S_2$  consider words  $w_s, v_t$  such that  $\varphi(s) = w_s, \psi(t) = v_t$ . Let

$$U_1 = \{s^{-1}w_s : s \in S_1\}, \quad U_2 = \{t^{-1}v_t : t \in S_2\}$$

We consider the presentation:

$$\langle S_1 \cup S_2 | R_1 \cup R_2 \cup U_1 \cup U_2 \rangle$$

We claim that there is a finite sequence of Tietze transformations from  $\langle S_1|R_1 \rangle$  to this presentation. Indeed using T2 we may introduce one by one the generators of  $S_2$  and the relations  $U_2$ . So we obtain the presentation

$$\langle S_1 \cup S_2 | R_1 \cup U_2 \rangle$$

The Tietze transformations give as an isomorphism

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$$\rho : \langle S_1 \cup S_2 | R_1 \cup U_2 \rangle \rightarrow \langle S_1 | R_1 \rangle$$

where  $\rho(s) = s$ ,  $\rho(t) = v_t$  for  $s \in S_1$ ,  $t \in S_2$ . We remark that  $\varphi \circ \rho$  is a homomorphism from  $\langle S_1 \cup S_2 | R_1 \cup U_2 \rangle$  to  $\langle S_2 | R_2 \rangle$  and  $\varphi \circ \rho(t) = t$  for all  $t \in S_2$ . It follows that for any  $r \in R_2$ ,  $\varphi \circ \rho(r) = r = 1$ , hence  $R_2 \subseteq \langle\langle R_1 \cup U_2 \rangle\rangle$ . So using T1 we obtain the presentation

$$\langle S_1 \cup S_2 | R_1 \cup R_2 \cup U_2 \rangle$$

We remark now that  $\varphi \circ \rho$  is still defined on this presentation and  $\varphi \circ \rho(s) = w_s$  for all  $s \in S_1$ , while  $\varphi \circ \rho(w_s) = w_s$ . It follows that  $s^{-1}w_s$  is mapped to 1 by  $\varphi \circ \rho$ , hence the relators  $U_1$  also follow from  $R_1 \cup R_2 \cup U_2$ . So applying T1 we obtain

$$\langle S_1 \cup S_2 | R_1 \cup R_2 \cup U_1 \cup U_2 \rangle$$

Similarly we see that there is a finite sequence of Tietze transformations from  $\langle S_2 | R_2 \rangle$  to this presentation. □

*Remark 3.3.* Let  $G = \langle X \rangle = \langle Y \rangle$  and  $X$  is finite then there is a finite subset  $Y' \subset Y$  such that  $G = \langle Y' \rangle$ .

So, in a sense, finite generation does not depend on the generating set we pick. The next proposition shows that something similar holds for finite presentability.

**Proposition 3.2.** *Let  $G \cong \langle S | R \rangle \cong \langle X | Q \rangle$  where  $S, X, R$  are finite. Then there is a finite subset  $Q'$  of  $Q$  such that  $G \cong \langle X | Q' \rangle$*

*Proof.* Let  $\varphi : F(S)/\langle\langle R \rangle\rangle \rightarrow F(X)/\langle\langle Q \rangle\rangle$  be an isomorphism. Let

$$S = \{s_1, \dots, s_n\}, R = \{r_1, \dots, r_k\}, X = \{x_1, \dots, x_m\}$$

Then the  $r_i$ 's are words in the  $s_j$ 's,  $r_i = r_i(s_1, \dots, s_n)$ . Let  $\varphi(s_i) = s'_i$ ,  $i = 1, \dots, n$ . If we see the  $s'_i$  as elements of  $F(X)$ , since  $\varphi$  is onto we have that the generators of  $G$  can be written in terms of the  $s'_i$ , so there are words  $w_j(s'_1, \dots, s'_n)$ ,  $j = 1, \dots, m$  and  $u_1, \dots, u_m \in \langle\langle Q \rangle\rangle$  such that

$$x_j = w_j(s'_1, \dots, s'_n)u_j, j = 1, \dots, m$$

where the equality is in  $F(X)$ . Since  $\varphi$  is a homomorphism we have also that

$$r_i(s'_1, \dots, s'_n) = v_i \in \langle\langle Q \rangle\rangle, i = 1, \dots, k$$

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Let  $Q'$  be a finite subset of  $Q$  such that all  $u_j, v_i, j = 1, \dots, m, i = 1, \dots, k$  can be written as products of conjugates of elements of  $Q'$ . We claim that  $\langle\langle Q' \rangle\rangle = \langle\langle Q \rangle\rangle$ . Indeed the map

$$\psi : F(S)/\langle\langle R \rangle\rangle \rightarrow F(X)/\langle\langle Q' \rangle\rangle$$

given by  $\psi(s_i) = s'_i$  is an onto homomorphism and  $\varphi = \pi \circ \psi$  where  $\pi$  is the natural quotient map

$$\pi : F(X)/\langle\langle Q' \rangle\rangle \rightarrow F(X)/\langle\langle Q \rangle\rangle$$

However  $\varphi$  is 1-1 so  $\pi$  is also 1-1. It follows that  $\langle\langle Q' \rangle\rangle = \langle\langle Q \rangle\rangle$ . □

**Proposition 3.3.** *If the word problem is solvable for the finite presentation  $\langle S|R \rangle$  of a group  $G$  then it is solvable for any other finite presentation  $\langle X|Q \rangle$  of  $G$ . The same is true for the conjugacy problem.*

*Proof.* To solve the word problem, given a word  $w$  on  $X$  we run ‘in parallel’ two procedures:

1) We list all elements in  $\langle\langle Q \rangle\rangle$ .

2) We list all homomorphisms  $\varphi : F(X)/\langle\langle Q \rangle\rangle \rightarrow F(S)/\langle\langle R \rangle\rangle$ . To find  $\varphi$  we enumerate  $|X|$ -tuples of words in  $F(S)$  and we check for each such choice whether the relations  $Q$  are satisfied. We note that this is possible to do since the word problem is solvable in  $\langle S|R \rangle$ . Given a homomorphism  $\varphi$  we check whether  $\varphi(w) \neq 1$  (which is possible to do since the word problem is solvable in  $\langle S|R \rangle$ ).

Clearly one of the procedures 1,2 will terminate. We note that if the conjugacy problem is solvable for a group then the word problem is also solvable (why?). To solve the conjugacy problem, given two words  $w, v$  on  $X$  we argue similarly:

1a) We list all elements of the form  $gvg^{-1}w^{-1}$ .

1b) We list all elements in  $\langle\langle Q \rangle\rangle$  and check whether some element is equal to  $gvg^{-1}w^{-1}$  in  $F(X)$ .

2. We list all homomorphisms  $\varphi : F(X)/\langle\langle Q \rangle\rangle \rightarrow F(S)/\langle\langle R \rangle\rangle$  and, given a homomorphism  $f$ , we check whether  $f(w), f(v)$  are not conjugate in  $\langle S|R \rangle$ . Clearly if  $f(w), f(v)$  are not conjugate in  $\langle S|R \rangle$  they are not conjugate in  $\langle X|Q \rangle$ . □

We remark that this proposition shows that the solvability of the word and the conjugacy problem is a property of the group and not of the presentation.

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