PHY 5207: GEOMETRIC GROUP THEORY
LECTURE 1

Introduction:

Geometric group theory is a descendant of combinatorial group theory, which in turn is the study of groups using their presentations. So one studies mainly infinite, finitely generated groups and is more interested in the class of finitely presented groups. Combinatorial group theory was developed in close connection to low dimensional topology and geometry.

The fundamental group of a compact manifold is finitely presented. So finitely presented groups give us an important invariant that helps us distinguish manifolds. Conversely topological techniques are often useful for studying groups. Dehn in 1912 posed some fundamental algorithmic problems for groups: The word problem, the conjugacy problem and the isomorphism problem. He solved these problems for fundamental groups of surfaces using hyperbolic geometry. Later the work of Dehn was generalized by Magnus and others, using combinatorial methods.

In recent years, due to the fundamental work of Stallings, Serre, Rips, Gromov powerful geometric techniques were introduced to the subject and combinatorial group theory developed closer ties with geometry and 3-manifold theory. This led to important results in 3-manifold theory and logic.

Some leitmotivs of combinatorial/geometric group theory are:

1. Solution of the fundamental questions of Dehn for larger classes of groups. One should remark that Novikov and Boone in the 50's showed that Dehn's problems are unsolvable in general. One may think of finitely presented groups as a jungle. The success of the theory is that it can deal with many natural classes of groups which are also important for topology/geometry. As we said the first attempts at this were combinatorial in nature, one imposed the so-called small cancelation conditions on the presentation. This was subsequently geometrized using van-Kampen diagrams by Lyndon-Schupp.

Gromov in 1987 used ideas coming from hyperbolic geometry to show that algorithmic problems can be solved for a very large ('generic') class of groups (called hyperbolic groups). It was Gromov's work that demonstrated that the geometric point of view was very fruitful for the study of groups and created geometric group theory. We will give a brief introduction to the theory of hyperbolic groups in the last sections of these notes.

2. One studies the structure of groups, in particular the subgroup structure. Ideally one would want to describe all subgroups of a given group. Some particular questions of interest are: existence of subgroups of finite index, existence of normal subgroups, existence of free subgroups and of free abelian subgroups etc.

Another structural question is the question of the decomposition of a group in 'simpler' groups. One would like to know if a group is a direct product, free product, amalgamated product etc. Further one would like to know if there is a canonical way to decompose a group in these types of products. The simplest example of such a theorem in the decomposition of a finitely generated abelian group as a direct product of cyclic groups.

In this course we will focus on an important tool of geometric group theory: the study of groups via their actions on trees, this is related to both structure theory and the subgroup structure of groups.

3. Construction of interesting examples of groups. Using amalgams and HNN extensions Novikov and Boone constructed finitely presented groups with unsolvable word and conjugacy problem. We mention also the Burnside question: Are there infinite finitely generated torsion groups? What about torsion groups of bounded exponent? The answer to both of these is yes (Novikov) but to this date it is not known whether there are infinite, finitely presented torsion groups.

Some of the recent notable successes of the theory is the solution of the Tarski problem by Sela and the solution of the virtually Haken conjecture and the virtually fibering conjecture by Agol-Wise.

The Tarski problem was an important problem in Logic asking whether the free groups of rank 2 and 3 have the same elementary theory i.e. whether the set of sentences which are valid in F_2 is the same with the set of sentences valid in F_3. Somewhat surprisingly the positive solution to this uses actions on Trees and Topology (and comprises more than 500 pages!).

The solution of the virtually Haken conjecture and the virtually fibering conjecture by Agol-Wise implies that every closed 3-manifold can be 'build' by gluing manifolds that are quite well understood topologically and after the fundamental work of Perelman completed our picture of what 3-manifolds look like. More explicitly an obvious way to construct 3-manifolds is by taking a product of a surface with $[0,1]$ and then gluing the two boundary surface pieces by a homeomorphism. The result of Agol-Wise shows that every 3-manifold can be build from pieces that have a finite sheeted cover that is either S^3 or of the form described in the previous sentence.

Free Groups

Definition 2.1. Let X be a subset of a group F. We say that F is a *free group with basis* X if any function φ from X to a group G can be extended uniquely to a homomorphism $\bar{\varphi} : F \to G$.

One may remark that the trivial group $\{e\}$ is a free group with basis the empty set. Also the infinite cyclic group $C =< a >$ is free with basis $\{a\}$. Indeed if G is any group and if $\varphi(a) = g$ then φ is extended to a homomorphism by

$$\bar{\varphi}(a^n) = \varphi(a)^n, \ \ n \in \mathbb{Z}$$

It is clear that this extension is unique. So $\{a\}$ is a free basis of C. We remark that $\{a^{-1}\}$ is another free basis of C.

Proposition 2.1. *Let X be a set. Then there is a free group $F(X)$ with basis X.*

Proof. We consider the set $S = X \sqcup X^{-1}$ where $X^{-1} = \{s^{-1} : s \in X\}$. A word in X is a finite sequence $(s_1, ..., s_n)$ where $s_i \in S$. We denote by e the empty sequence. We usually denote words as strings of letters, so eg if (a, a^{-1}, b, b) is a word we write simply $aa^{-1}bb$ or $aa^{-1}b^2$. Let W be the set of words in S. We define an equivalence relation $\sim$ on W generated by:

$$uaa^{-1}v \sim uv, \ ua^{-1}av \sim uv \quad \text{for any } a \in S, \, u, v \in W$$

So two words are equivalent if we can go to one from the other by a finite sequence of insertions and/or deletions of consecutive inverse letters.

Let $F := W/\sim$ be the set of equivalence classes of this relation. We denote by $[w]$ the equivalence class of $w \in W$. If

$$w = (a_1, ..., a_n), \ v = (b_1, ..., b_k)$$

then we define the product wv of w, v by

$$wv = (a_1, ..., a_n, b_1, ..., b_k)$$

We remark that if $w_1 \sim w_2$, $v_1 \sim v_2$ then $w_1v_1 \sim w_2v_2$, so we define multiplication on F by $[w][v] = [wv]$. We claim that F with this operation is a group. Indeed $e = [\emptyset]$ is the identity element and if $w = (b_1, ..., b_n)$ the inverse element is given by $w^{-1} = (b_n^{-1}, ..., b_1^{-1})$. Here we follow the usual convention that if $s^{-1} \in X^{-1}$ then $(s^{-1})^{-1} = s$. It is clear that associativity holds:

$$([w][u])[v] = [w]([u][v])$$

since both sides are equal to $[wuv]$.

If $w \in W$ we denote by $|w|$ the length of w (eg $|aa^{-1}ba| = 4$). We say that a word w is *reduced* if it does not contain a subword of the form aa^{-1} or $a^{-1}a$ where $a \in X$. To complete the proof of the theorem we need the following:

Lemma 2.1. *Every equivalence class $[w] \in F$ has a unique representative which is a reduced word.*

Proof. It is clear that $[w]$ contains a reduced word. Indeed one starts with w and eliminates successively pairs of the form aa^{-1} or $a^{-1}a$ till none are left. What this lemma says is that the order under which eliminations are performed doesn't matter. This is quite obvious but we give here a formal (and rather tedious) argument.

It is enough to show that two distinct reduced words w, v are not equivalent. We argue by contradiction. If w, v are equivalent then there is a sequence

$$w_0 = w, w_1, ..., w_n = v$$

where each w_{i+1} is obtained from w_i by insertion or deletion of a pair of the form aa^{-1} or $a^{-1}a$. We assume that for the sequence w_i the sum of the lengths $L = \sum |w_i|$ is the minimal possible among all sequences of this type going from w to v. Since w, v are reduced we have that $|w_1| > |w_0|$, $|w_{n-1}| > |w_n|$. It follows that for some i we have

$$|w_i| > |w_{i-1}|, \ |w_i| > |w_{i+1}|$$

So w_{i-1} is obtained from w_i by deletion of a pair a, a^{-1} and w_{i+1} is obtained from w_i by deletion of a pair b, b^{-1}. If these two pairs are distinct in w_i then we can delete b, b^{-1} first and then add a, a^{-1} decreasing L. More precisely if we have for instance

$$w_i = u_1bb^{-1}u_2aa^{-1}u_3, \ w_{i-1} = u_1bb^{-1}u_2u_3, \ w_{i+1} = u_1u_2aa^{-1}u_3$$

we can replace w_i by $u_1u_2u_3$. In this way L decreases by 4, which is a contradiction.

Now if the pairs a, a^{-1}, b, b^{-1} are not distinct we remark that $w_{i-1} = w_{i+1}$ which is again a contradiction. □

We can now identify X with the subset $\{[s] : s \in X\}$ of F. Let G be a group and let $\varphi : X \to G$ be any function. Then we define a homomorphism $\bar{\varphi} : F \to G$ as follows: if $s^{-1} \in X^{-1}$ we define $\bar{\varphi}(s^{-1}) = \varphi(s)^{-1}$. If $w = s_1 ... s_n$ is a reduced word we define

$$\bar{\varphi}([w]) = \varphi(s_1)...\varphi(s_n)$$

It is easy to see that $\bar{\varphi}$ is a homomorphism. We remark finally that this extension of φ is unique by definition. So $F(X) = F$ is a free group with basis X. □

Using the lemma above we can identify the elements of F with the reduced words of W.

Remark 2.1. In the sequel if w is any word in X (not necessarily reduced) we will also consider w as an element of the free group $F(X)$. This could cause some confusion as it is possible to have $w \neq v$ as words but $w = v$ in $F(X)$.

Corollary 2.1. *Every group is a quotient group of a free group.*

Proof. Let G be a group. We consider the free group with basis G, $F(G)$. If $\varphi : G \to G$ is the identity map $\varphi(g) = g$, then φ can be extended to an epimorphism $\bar{\varphi} : F(G) \to G$. If $N = ker(\bar{\varphi})$ then

$$G \cong F(G)/N$$

□

If X is a set we denote by $|X|$ the cardinality of X.

Proposition 2.2. *Let $F(X), F(Y)$ be free groups on X, Y. Then $F(X)$ is isomorphic to $F(Y)$ if and only if $|X| = |Y|$.*

Proof. Assume that $|X| = |Y|$. We consider a 1-1 and onto function $f : X \to Y$. Let $h = f^{-1}$. The maps f, h are extended to homomorphisms $\bar{f}, \bar{h}$ and $\bar{f} \circ \bar{h}$ is the identity on $F(Y)$ while $\bar{h} \circ \bar{f}$ is the identity on $F(X)$ so $\bar{f}$ is an isomorphism.

Conversely assume that $F(X)$ is isomorphic to $F(Y)$. If X, Y are infinite sets then the cardinality of $F(X)$, $F(Y)$ is equal to the cardinality, respectively of X, Y. So if these groups are isomorphic $|X| = |Y|$. Otherwise if, say, $|X|$ is finite, we note that there are $2^{|X|}$ homomorphisms from $F(X)$ to $\mathbb{Z}_2$. Since $F(X) \cong F(Y)$ we have that $2^{|X|} = 2^{|Y|}$ so $|X| = |Y|$. □

Remark 2.2. Let $F(X)$ be a free group on X. If A is any set of generators of $F(X)$ then $|A| \geq |X|$.

Indeed if $|A| < |X|$ then there are at most $2^{|A|}$ homomorphisms from $F(X)$ to $\mathbb{Z}_2$, a contradiction.

If F is a free group with free basis X then the *rank* of F is the cardinality of X. We denote by F_n the free group of rank n.

The word problem

If F is a free group with free basis X then we identify the elements of F with the words in X. This is a bit ambiguous as equivalent words represent the same element. The word problem in this case is to decide whether a word represents the identity element. This is of course trivial as it amounts to checking whether the word reduces to the empty word after cancelations.

The conjugacy problem

Definition 2.2. If $w = s_1 ... s_n$ is a word then the cyclic permutations of w are the words:

$$s_n s_1 ... s_{n-1},\ s_{n-1} s_n ... s_{n-2},\ .....,\ s_2 ... s_n s_1$$

A word is called *cyclically reduced* if it is reduced and all its cyclic permutations are reduced words.

We remark that a word w on S is cyclically reduced if w is reduced and $w \neq xvx^{-1}$ for any $x \in S \sqcup S^{-1}$.

Proposition 2.3. *Let $F(X)$ be a free group. Every word $w \in F(X)$ is conjugate to a cyclically reduced word. Two cyclically reduced words w, v are conjugate if and only if they are cyclic permutations of each other.*

Proof. Let r be a word of minimal length that is conjugate to w. If $r = xux^{-1}$ then r is conjugate to u and $|u| < |r|$ which is a contradiction. Hence r is cyclically reduced.

Let w now be a cyclically reduced word. Clearly every cyclic permutation of w is conjugate to w. We show that a cyclically reduced word conjugate to w is a cyclic permutation of w. We argue by contradiction.

Let g be a word of minimal length such that the reduced word v representing $g^{-1}wg$ is cyclically reduced but is not a cyclic permutation of w. If the word gvg^{-1} is reduced then it is not cyclically reduced. But $w = gvg^{-1}$ and w is cyclically reduced so gvg^{-1} is not reduced. If $g = s_1 ... s_n$, $s_i \in X \cup X^{-1}$ then either $v = s_n^{-1} u$ or $v = us_n$. If $v = s_n^{-1} u$ then

$$gvg^{-1} = s_1 ... s_{n-1}(us_n^{-1})(s_1 ... s_{n-1})^{-1}$$

By our assumption that g is minimal length we have that us_n^{-1} is a cyclic permutation of w. But then $v = s_n^{-1} u$ is also a cyclic permutation of w. We argue similarly if $v = us_n$. □

Using this proposition it is easy to solve algorithmically the conjugacy problem in a free group.

Remark 2.3. A word g is cyclically reduced if and only if gg is reduced. Clearly if a word w is reduced then $w = uvu^{-1}$ where v is cyclically reduced.

Proposition 2.4. *A free group F has no elements of finite order.*

Proof. Let $g \in F$. Then g is conjugate to a cyclically reduced word h. Clearly g, h have the same order. We remark now that h^n is reduced for any $n \in \mathbb{N}$ so $h^n \neq e$, ie the order of g is infinite. □

Proposition 2.5. *Let F be a free group and $g, h \in F$. If $g^k = h^k$ for some $k \geq 1$ then $g = h$.*

Proof. Let's say that $g = ug_1u^{-1}$ with $u \in F$ and g_1 cyclically reduced. Then $g_1^k = (u^{-1}gu)^k = (u^{-1}hu)^k$. Let h_1 be the reduced word equal to $u^{-1}hu$.

If h_1 is not cyclically reduced then $g_1^k \neq h_1^k$ since h_1^k is not cyclically reduced. Otherwise

$$g_1^k = h_1^k \implies g_1 = h_1$$

since g_1^k, h_1^k are reduced words. Hence $g = h$. □

Exercises 2.1. 1. Show that F_2 has a free subgroup of rank 3.

2. Show that F_2 has a free subgroup of infinite rank.