PHY 5207: GEOMETRIC GROUP THEORY
ASSIGNMENT 1

ATTEMPT ALL QUESTIONS

1. Let $\langle S|R\rangle$ be a finite presentation of a group G.

i. Explain how to enumerate all words on S representing the identity in G.

ii. Explain how to enumerate all finite presentations of G.

2. Let $\langle S|R\rangle$ be a finite presentation of a finite group G. Give an algorithm to solve the word problem for this presentation.

3. Show that if a finitely presented group $G = \langle S|R\rangle$ has a solvable word problem and the finitely presented group $H = \langle S'|R'\rangle$ is isomorphic to some subgroup of G then H also has a solvable word problem. Note that here we *do not assume* that we are given an injective homomorphism $f : H \to G$.

4. If H is a finitely generated subgroup of G then the *membership problem* for H asks whether there is an algorithm to decide if $g \in G$ lies in H. Show that the membership problem is solvable for cyclic subgroups of F_n (the free group of rank n). In other words there is an algorithm such that given $u, w \in F_n$ decides whether $u \in < w >$.

5. An infinite finitely generated group is called just infinite if all its quotients are finite groups. Show that every infinite finitely generated group has a quotient that is just infinite.

6. Let H be a finite index subgroup of G. Show that there is a normal finite index subgroup of G, N such that $N \subset H$. Deduce that G is residually finite iff for every $g \in G$ there is some finite index subgroup of G, H, such that $g \notin H$.

7. Let G be a finitely generated group. Show that G has finitely many subgroups of index n. (*hint:* use the previous exercise).

8. Show that if G has a finite index subgroup which is residually finite then G itself is residually finite.

9. Let G be a residually finite group. Show that if G has finitely many conjugacy classes of elements of finite order then G has a torsion free finite index subgroup.

10. Give an example of a residually finite group which is not Hopf.