PHY 5207: GEOMETRIC GROUP THEORY
MID-TERM EXAM

ATTEMPT ALL QUESTIONS (10 MARKS EACH)

1. We say that a subgroup H of G is *separable* if it is equal to the intersection of all finite index subgroups of G containing it.

i) Show that G is residually finite if and only if $\{e\}$ is a separable subgroup of G.

ii) Let G be residually finite and let $r : G \to G$ be a retract, that is a homomorphism such that $r^2 = r$. Show that $r(G)$ is a separable subgroup.

2. Show that every cyclic subgroup of F_n (the free group of rank n) is separable.

3. If H is a subgroup of the free group F_n of index $|F_n : H| = r$ show that H is a free group of rank $r(n-1)+1$. (*hint:* look closely at the proof that H is free).

4. If $g \neq 1$ is an element of F_n show that the normalizer of $< g >$ in F_n is a cyclic group.(*hint:* if u is in the normalizer then $< u, g >$ is free.)

5. Determine the center of the group $\langle a, b | a^2 = b^3 \rangle$.

6. Show that a finite group H acting on a tree T either fixes a vertex of T or fixes a geometric edge of T (ie $H \cdot e \subset \{e, \bar{e}\}$ for some edge e). Deduce that any finite subgroup of an amalgam $A *_C B$ is contained in a conjugate of A or B.

7. Show that if A, B are residually finite then $A * B$ is also residually finite.

8. Assume that $G = A *_C B$. Show that if G, C are finitely generated then A, B are also finitely generated.

9. Assume that $G = A *_C B$ with A, B, C finitely presented. Show that if the word problem of A, B is decidable and if the membership problem for C in A, B is also decidable then the word problem of G is decidable.

10. The fundamental group of a surface group of genus 2 has a presentation:

$$G = \langle a, b, c, d | [a, b] = [c, d] \rangle$$

where we denote by $[a, b]$ the commutator: $aba^{-1}b^{-1}$. Show that G is an amalgam of two free groups over $\mathbb{Z}$. Deduce that the word problem of G is decidable.