PHY 5207: GEOMETRIC GROUP THEORY
FINAL EXAM

ATTEMPT ALL QUESTIONS (10 MARKS EACH)

1. Show that if $G = A *_C B$ and $|A : C| \geq 3$, $|B : C| \geq 2$ then G has a free subgroup of rank 2.

2. i) Let G be a finitely generated group such that $G = A *_C B$ where $|A : C| = 2, |B : C| = 2$, and A, B are finite. Show that G has a finite index subgroup isomorphic to $\mathbb{Z}$.

ii) Show that if $G = A*_A$ with A finite then G has a finite index subgroup isomorphic to $\mathbb{Z}$.

3. Show that every finitely generated subgroup of $F_n *_{<c>} F_n$ (F_n free of rank n) is finitely presented. (*hint:* it is enough to show that the subgroup is the fundamental group of a *finite* graph of groups with cyclic edge groups).

4. Let G be a finitely presented group. Show that the HNN-extension $G \underset{A}{*}$ is finitely presented if and only if A is finitely generated.

5. Let G be a group acting on a tree T without inversions.

Show that if $g \in G$ fixes no vertex of T then there is a line $L \subset T$ such that g acts on L by translations. (*hint:* Consider the vertices for which $d(v, gv)$ is minimum). Show that if $h = aga^{-1}$ then h fixes no vertex of T and acts on $a(L)$ by translations.

6. Show that the product of two free groups of rank 2, $F_2 \times F_2$ can not be written as a non trivial amalgam over $\mathbb{Z}$. (*hint:* If g, h commute and do not fix a vertex then they translate along the same line. Use this to show that any action on a tree has 'big' edge stabilizers.)

7. Show that the group $G = \mathbb{Z}^2 * \mathbb{Z}^2$ can be written as an HNN-extension over $\mathbb{Z}$. Show that G can be written non-trivially as the fundamental group of a graph of groups with 3 edges.

8. Let $H = \pi_1(G, Y, a_0)$. Show that if Y is not a tree then H admits an epimorphism onto $\mathbb{Z}$.