MTH 401 – DIFFERENTIAL EQUATIONS

LECTURE 2: SEPARABLE EQUATIONS

The differential equation of the form

$$\frac{dy}{dx} = f(x, y)$$

is called **separable** if it can be written in the form

$$\frac{dy}{dx} = h(x)g(y)$$

To solve a separable equation, we perform the following steps:

1. We solve the equation $g(y) = 0$ to find the constant solutions of the equation.

2. For non-constant solutions we write the equation in the form.

$$\frac{dy}{g(y)} = h(x)dx$$

Then integrate $\int \frac{1}{g(y)} dy = \int h(x)dx$

to obtain a solution of the form

$$G(y) = H(x) + C$$

3. We list the entire constant and the non-constant solutions to avoid repetition..

4. If you are given an IVP, use the initial condition to find the particular solution.

Note that:

(a) No need to use two constants of integration because $C_1 - C_2 = C$.

(b) The constants of integration may be relabeled in a convenient way.

(c) Since a particular solution may coincide with a constant solution, **step 3 is important.**

Example 1:

Find the particular solution of

$$\frac{dy}{dx} = \frac{y^2 - 1}{x}, \quad y(1) = 2$$

Solution:

1. By solving the equation

$$y^2 - 1 = 0$$

We obtain the constant solutions

$$y = \pm 1$$

2. Rewrite the equation as

$$\frac{dy}{y^2 - 1} = \frac{dx}{x}$$

Resolving into partial fractions and integrating, we obtain

$$\frac{1}{2}\int\left[\frac{1}{y-1} - \frac{1}{y+1}\right]dy = \int \frac{1}{x}dx$$

Integration of rational functions, we get

$$\frac{1}{2}\ln\frac{|y-1|}{|y+1|} = \ln|x| + C$$

3. The solutions to the given differential equation are

$$\begin{cases} \frac{1}{2}\ln\frac{|y-1|}{|y+1|} = \ln|x| + C \\ y = \pm 1 \end{cases}$$

4. Since the constant solutions do not satisfy the initial condition, we plug in the condition

$y = 2$ When $x = 1$ in the solution found in step 2 to find the value of C.

$$\frac{1}{2}\ln\left(\frac{1}{3}\right) = C$$

The above implicit solution can be rewritten in an explicit form as**:**

$$y = \frac{3 + x^2}{3 - x^2}$$

Example 2:

Solve the differential equation

$$\frac{dy}{dt} = 1 + \frac{1}{y^2}$$

Solution:

1. We find roots of the equation to find constant solutions

$$1 + \frac{1}{y^2} = 0$$

No constant solutions exist because the equation has no real roots.

2. For non-constant solutions, we separate the variables and integrate

$$\int \frac{dy}{1 + 1/y^2} = \int dt$$

Since $$\frac{1}{1+1/y^2}=\frac{y^2}{y^2+1}=1-\frac{1}{y^2+1}$$

Thus $$\int\frac{dy}{1+1/y^2}=y-\tan^{-1}(y)$$

So that $$y-\tan^{-1}(y)=t+C$$

It is **not easy** to find the **solution in an explicit form** i.e. y as a function of t.

3. Since ∃ no constant solutions, all solutions are given by the implicit equation found
in step 2.

Example 3:

Solve the initial value problem

$$\frac{dy}{dt}=1+t^2+y^2+t^2y^2,\quad y(0)=1$$

Solution:

1. Since $$1+t^2+y^2+t^2y^2=(1+t^2)(1+y^2)$$
   The equation is separable & has no constant solutions because ∃ no real roots of
   $$1+y^2=0.$$
2. For non-constant solutions we separate the variables and integrate.
   $$\frac{dy}{1+y^2}=(1+t^2)dt$$
   $$\int\frac{dy}{1+y^2}=\int(1+t^2)dt$$
   $$\tan^{-1}(y)=t+\frac{t^3}{3}+C$$
   Which can be written as
   $$y=\tan\left(t+\frac{t^3}{3}+C\right)$$
3. Since ∃ no constant solutions, all solutions are given by the implicit or explicit equation.

4. The initial condition $y(0)=1$ gives

$$C=\tan^{-1}(1)=\frac{\pi}{4}$$

The particular solution to the initial value problem is

$$\tan^{-1}(y) = t + \frac{t^3}{3} + \frac{\pi}{4}$$

or in the explicit form $\quad y = \tan\left(t + \frac{t^3}{3} + \frac{\pi}{4}\right)$

Example 4:

Solve

$$(1+x)dy - ydx = 0$$

Solution:

Dividing with $(1+x)y$, we can write the given equation as

$$\frac{dy}{dx} = \frac{y}{(1+x)}$$

1. The only constant solution is $y = 0$
2. For non-constant solution we separate the variables

$$\frac{dy}{y} = \frac{dx}{1+x}$$

Integrating both sides, we have

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln|y| = \ln|1+x| + c_1$$

$$y = e^{\ln|1+x|+c_1} = e^{\ln|1+x|}.e^{c_1}$$

or $\quad y = |1+x|\, e^{c_1} == \pm e^{c_1}(1+x)$

$$y = C(1+x), \quad C = \pm e^{c_1}$$

If we use $\ln|c|$ instead of c_1 then the solution can be written as

$$\ln|y| = \ln|1+x| + \ln|c|$$

or $\quad \ln|y| = \ln|c(1+x)|$

So that $\quad y = c(1+x).$

3. The solutions to the given equation are

$$y = c(1+x)$$

$$y = 0$$

Example 5

Solve

$$xy^4dx + (y^2+2)e^{-3x}dy = 0.$$

Solution:

The differential equation can be written as

$$\frac{dy}{dx} = \left(-xe^{3x}\right)\left(\frac{y^4}{y^2+2}\right)$$

1. Since $\frac{y^4}{y^2+2} \Rightarrow y = 0$. Therefore, the only constant solution is $y = 0$.

2. We separate the variables

$$xe^{3x}dx + \frac{y^2+2}{y^4}dy = 0 \quad \text{or} \quad xe^{3x}dx + \left(y^{-2}+2y^{-4}\right)dy = 0$$

Integrating, with use integration by parts by parts on the first term, yields

$$\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} - y^{-1} - \frac{2}{3}y^{-3} = c_1$$

$$e^{3x}(3x-1) = \frac{9}{y} + \frac{6}{y^3} + c \quad \text{where} \quad 9c_1 = c$$

3. All the solutions are

$$e^{3x}(3x-1) = \frac{9}{y} + \frac{6}{y^3} + c$$

$$y = 0$$

Example 6:

Solve the initial value problems

(a) $\frac{dy}{dx} = (y-1)^2, \quad y(0) = 1$ (b) $\frac{dy}{dx} = (y-1)^2, \quad y(0) = 1.01$

and compare the solutions.

Solutions:

1. Since $(y-1)^2 = 0 \Rightarrow y = 1$. Therefore, the only constant solution is $y = 0$.

2. We separate the variables

$$\frac{dy}{(y-1)^2} = dx \quad \text{or} \quad (y\text{-}1)^{-2}\,dy = dx$$

Integrating both sides we have

$$\int (y-1)^{-2}\,dy = \int dx$$

$$\frac{(y-1)^{-2+1}}{-2+1} = x + c$$

or

$$-\frac{1}{y-1} = x + c$$

3. All the solutions of the equation are

$$-\frac{1}{y-1} = x + c$$

$$y = 1$$

4. We plug in the conditions to find particular solutions of both the problems

(a) $y(0) = 1 \Rightarrow y = 1$ when $x = 0$. So we have

$$-\frac{1}{1-1} = 0 + c \Rightarrow c = -\frac{1}{0} \Rightarrow c = -\infty$$

The particular solution is

$$-\frac{1}{y-1} = -\infty \Rightarrow y - 1 = 0$$

So that the solution is $y = 1$, which is same as constant solution.

(b) $y(0) = 1.01 \Rightarrow y = 1.01$ when $x = 0$. So we have

$$-\frac{1}{1.01-1} = 0 + c \Rightarrow c = -100$$

So that solution of the problem is

$$-\frac{1}{y-1} = x - 100 \Rightarrow y = 1 + \frac{1}{100 - x}$$

5. Comparison: A radical change in the solutions of the differential equation has Occurred corresponding to a very small change in the condition**!!**

Example 7:

Solve the initial value problems

(a) $\frac{dy}{dx} = (y-1)^2 + 0.01, \quad y(0) = 1$ (b) $\frac{dy}{dx} = (y-1)^2 - 0.01, \quad y(0) = 1.$

Solution:

(a) First consider the problem

$$\frac{dy}{dx} = (y-1)^2 + 0.01, \qquad y(0) = 1$$

We separate the variables to find the non-constant solutions

$$\frac{dy}{\left(\sqrt{0.01}\right)^2 + (y-1)^2} = dx$$

Integrate both sides

$$\int \frac{d(y-1)}{\left(\sqrt{0.01}\right)^2 + (y-1)^2} = \int dx$$

So that
$$\frac{1}{\sqrt{0.01}} \tan^{-1} \frac{y-1}{\sqrt{0.01}} = x + c$$

$$\tan^{-1}\left(\frac{y-1}{\sqrt{0.01}}\right) = \sqrt{0.01}(x+c)$$

$$\frac{y-1}{\sqrt{0.01}} = \tan\left[\sqrt{0.01}(x+c)\right]$$

or
$$y = 1 + \sqrt{0.01}\tan\left[\sqrt{0.01}(x+c)\right]$$

Applying $y(0) = 1 \Rightarrow y = 1$ when $x = 0$, we have

$$\tan^{-1}(0) = \sqrt{0.01}(0+c) \Rightarrow 0 = c$$

Thus the solution of the problem is

$$y = 1 + \sqrt{0.01}\tan\left(\sqrt{0.01}\, x\right)$$

(b) Now consider the problem

$$\frac{dy}{dx} = (y-1)^2 - 0.01, \qquad y(0) = 1.$$

We separate the variables to find the non-constant solutions

$$\frac{dy}{(y-1)^2-\left(\sqrt{0.01}\right)^2}=dx$$

$$\int\frac{d(y-1)}{(y-1)^2-\left(\sqrt{0.01}\right)^2}=\int dx$$

$$\frac{1}{2\sqrt{0.01}}\ln\left|\frac{y-1-\sqrt{0.01}}{y-1+\sqrt{0.01}}\right|=x+c$$

Applying the condition $y(0)=1\Rightarrow y=1$ when $x=0$

$$\frac{1}{2\sqrt{0.01}}\ln\left|\frac{-\sqrt{0.01}}{\sqrt{0.01}}\right|=c\Rightarrow c=0$$

$$\ln\left|\frac{y-1-\sqrt{0.01}}{y-1+\sqrt{0.01}}\right|=2\sqrt{0.01}\,x$$

$$\frac{y-1-\sqrt{0.01}}{y-1+\sqrt{0.01}}=\frac{e^{2\sqrt{0.01}\,x}}{1}$$

Simplification:

By using the property $\frac{a}{b}=\frac{c}{d}\Rightarrow\frac{a+b}{a-b}=\frac{c+d}{c-d}$

$$\frac{y-1-\sqrt{0.01}+y-1+\sqrt{0.01}}{y-1-\sqrt{0.01}-y+1-\sqrt{0.01}}=\frac{e^{2\sqrt{0.01}x}+1}{e^{2\sqrt{0.01}x}-1}$$

$$\frac{2y-2}{-2\sqrt{0.01}}=\frac{e^{2\sqrt{0.01}}+1}{e^{2\sqrt{0.01}}-1}$$

$$\frac{y-1}{-\sqrt{0.01}}=\frac{e^{2\sqrt{0.01}}+1}{e^{2\sqrt{0.01}}-1}$$

$$y-1=-\sqrt{0.01}\left(\frac{e^{2\sqrt{0.01}}+1}{e^{2\sqrt{0.01}}-1}\right)$$

$$y=1-\sqrt{0.01}\left(\frac{e^{2\sqrt{0.01}}+1}{e^{2\sqrt{0.01}}-1}\right)$$

Comparison:

The solutions of both the problems are

(a) $y = 1+\sqrt{0.01}\ \tan\left(\sqrt{0.01}\,x\right)$

(b) $y = 1-\sqrt{0.01}\left(\dfrac{e^{2\sqrt{0.01}}+1}{e^{2\sqrt{0.01}}-1}\right)$

Again a radical change has occurred corresponding to a very small in the differential equation!

Exercise:

Solve the given differential equation by separation of variables.

1. $\dfrac{dy}{dx}=\left(\dfrac{2y+3}{4x+5}\right)^2$

2. $\sec^2 x dy+\csc y dx=0$

3. $e^y \sin 2x dx+\cos x\left(e^{2y}-y\right)dy=0$

4. $\dfrac{dy}{dx}=\dfrac{xy+3x-y-3}{xy-2x+4y-8}$

5. $\dfrac{dy}{dx}=\dfrac{xy+2y-x-2}{xy-3y+x-3}$

6. $y\left(4-x^2\right)^{\frac{1}{2}}dy=\left(4+y^2\right)^{\frac{1}{2}}dx$

7. $\left(x+\sqrt{x}\right)\dfrac{dy}{dx}=y+\sqrt{y}$

Solve the given differential equation subject to the indicated initial condition.

8. $\left(e^{-y}+1\right)\sin x dx = \left(1+\cos x\right)dy$ **,** $y(0)=0$

9. $\left(1+x^{4}\right)dy + x\left(1+4y^{2}\right)dx = 0$ **,** $y(1)=0$

10. $ydy = 4x\left(y^{2}+1\right)^{\frac{1}{2}} dx$ **,** $y(0)=1$