MTH 401 – DIFFERENTIAL EQUATIONS

LECTURE 3: HOMOGENOUS DIFFERENTIAL EQUATIONS

A differential equation of the form

$$\frac{dy}{dx} = f(x,y)$$

Is said to be *homogeneous* if the function $f(x,y)$ is homogeneous, which means

$f(tx,ty) = t^n f(x,y)$ For some real number n, for any number t.

Example 1

Determine whether the following functions are homogeneous

$$\begin{cases} f(x,y) = \dfrac{xy}{x^2+y^2} \\ g(x,y) = \ln\left(-3x^2y/(x^3+4xy^2)\right) \end{cases}$$

Solution:

The functions $f(x,y)$ is homogeneous because

$$f(tx,ty) = \frac{t^2xy}{t^2(x^2+y^2)} = \frac{xy}{x^2+y^2} = f(x,y)$$

Similarly, for the function $g(x,y)$ we see that

$$g(tx,ty) = \ln\left(\frac{-3t^3x^2y}{t^3(x^3+4xy^2)}\right) = \ln\left(\frac{-3x^2y}{x^3+4xy^2}\right) = g(x,y)$$

Therefore, the second function is also homogeneous.

Hence the differential equations

$$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ \dfrac{dy}{dx} = g(x,y) \end{cases}$$

Are homogeneous differential equations

Method of Solution:

To solve the homogeneous differential equation

$$\frac{dy}{dx} = f(x, y)$$

We use the substitution

$$v = \frac{y}{x}$$

If $f(x, y)$ is homogeneous of degree zero, then we have

$$f(x, y) = f(1, v) = F(v)$$

Since $y' = xv' + v$, the differential equation becomes

$$x\frac{dv}{dx} + v = f(1, v)$$

This is a separable equation. We solve and go back to old variable y through $y = xv$.

Summary:

1. Identify the equation as homogeneous by checking $f(tx, ty) = t^n f(x, y)$;

2. Write out the substitution $v = \frac{y}{x}$;

3. Through easy differentiation, find the new equation satisfied by the new function v;

$$x\frac{dv}{dx} + v = f(1, v)$$

4. Solve the new equation (which is always separable) to find v;

5. Go back to the old function y through the substitution $y = vx$;

6. If we have an IVP, we need to use the initial condition to find the constant of integration.

Caution:

- Since we have to solve a separable equation, we must be careful about the constant solutions.
- If the substitution $y = vx$ does not reduce the equation to separable form then the equation is not homogeneous or something is wrong along the way.

Illustration:

Example 2 Solve the differential equation

$$\frac{dy}{dx}=\frac{-2x+5y}{2x+y}$$

Solution:

Step 1. It is easy to check that the function

$$f(x,y)=\frac{-2x+5y}{2x+y}$$

is a homogeneous function.

Step 2. To solve the differential equation we substitute

$$v=\frac{y}{x}$$

Step 3. Differentiating w.r.t x, we obtain

$$xv'+v=\frac{-2x+5xv}{2x+xv}=\frac{-2+5v}{2+v}$$

which gives

$$\frac{dv}{dx}=\frac{1}{x}\left(\frac{-2+5v}{2+v}-v\right)$$

This is a separable. At this stage please refer to the **Caution!**

Step 4. Solving by separation of variables all solutions are implicitly given by

$$-4\ln(|v-2|)+3\ln|v-1|=\ln(|x|)+C$$

Step 5. Going back to the function y through the substitution $y=vx$, we get

$$-4\ln|y-2x|+3\ln|y-x|=\ C$$

$$-4\ln\left|\frac{y-2x}{x}\right|+3\ln\left|\frac{y-x}{x}\right|=\ln|x|+c$$

$$\ln\left|\frac{y-2x}{x}\right|^{-4}+\ln\left|\frac{y-x}{x}\right|^{3}=\ln x+\ln c_1,\ c=\ln c_1$$

$$\ln\left|\frac{(y-2x)^{-4}}{x^{-4}}\right|+\ln\left|\frac{(y-x)^{3}}{x^{3}}\right|=\ln c_1 x$$

$$\ln\left|\frac{(y-2x)^{-4}}{x^{-4}}.\frac{(y-x)^{3}}{x^{3}}\right|=\ln c_1 x$$

$$\frac{(y-2x)^{-4}}{x^{-4}}.\frac{(y-x)^{3}}{x^{3}}=c_1 x$$

$$x(y-2x)^{-4}(y-x)^{3}=c_1 x$$

$$(y-2x)^{-4}(y-x)^{3}=c_1$$

Note that the implicit equation can be rewritten as

$$(y-x)^3=C_1(y-2x)^4.$$

Equations reducible to homogenous form

The differential equation

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

is not homogenous. However, it can be reduced to a homogenous form as detailed below

Case 1: $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

We use the substitution $z = a_1x + b_1y$ which reduces the equation to a separable equation in the variables x and z. Solving the resulting separable equation and replacing z with $a_1x + b_1y$, we obtain the solution of the given differential equation.

Case 2: $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

In this case we substitute

$$x = X + h, \qquad y = Y + k$$

Where h and k are constants to be determined. Then the equation becomes

$$\frac{dY}{dX} = \frac{a_1X + b_1Y + a_1h + b_1k + c_1}{a_2X + b_2Y + a_2h + b_2k + c_2}$$

We choose h and k such that

$$\left.\begin{aligned} a_1h + b_1k + c_1 = 0 \\ a_2h + b_2k + c_2 = 0 \end{aligned}\right\}$$

This reduces the equation to

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

Which is homogenous differential equation in X and Y, and can be solved accordingly. After having solved the last equation we come back to the old variables x and y.

Example 3

Solve the differential equation

$$\frac{dy}{dx} = -\frac{2x+3y-1}{2x+3y+2}$$

Solution:

Since $\frac{a_1}{a_2} = 1 = \frac{b_1}{b_2}$, we substitute $z = 2x + 3y$, so that

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dz}{dx} - 2\right)$$

Thus the equation becomes

$$\frac{1}{3}\left(\frac{dz}{dx} - 2\right) = -\frac{z-1}{z+2}$$

i.e. $$\frac{dz}{dx} = \frac{-z+7}{z+2}$$

This is a variable separable form, and can be written as

$$\left(\frac{z+2}{-z+7}\right)dz = dx$$

Integrating both sides we get

$$-z - 9\ln(z-7) = x + A$$

Simplifying and replacing z with $2x + 3y$, we obtain

$$-\ln(2x+3y-7)^9 = 3x + 3y + A$$

or $$(2x+3y-7)^{-9} = ce^{3(x+y)}, \qquad c = e^A$$

Example 4

Solve the differential equation

$$\frac{dy}{dx}=\frac{(x+2y-4)}{2x+y-5}$$

Solution:

By substitution

$$x=X+h, \qquad y=Y+k$$

The given differential equation reduces to

$$\frac{dY}{dX}=\frac{(X+2Y)+(h+2k-4)}{(2X+Y)+(2h+k-5)}$$

We choose h and k such that

$$h+2k-4=0, \quad 2h+k-5=0$$

Solving these equations we have $h=2,\ k=1$. Therefore, we have

$$\frac{dY}{dX}=\frac{X+2Y}{2X+Y}$$

This is a homogenous equation. We substitute $Y=VX$ to obtain

$$X\frac{dV}{dX}=\frac{1-V^2}{2+V} \quad \text{or} \quad \left[\frac{2+V}{1-V^2}\right]dV=\frac{dX}{X}$$

Resolving into partial fractions and integrating both sides we obtain

$$\int\left[\frac{3}{2(1-V)}+\frac{1}{2(1+V)}\right]dV=\int\frac{dX}{X}$$

or

$$-\frac{3}{2}\ln(1-V)+\frac{1}{2}\ln(1+V)=\ln X+\ln A$$

Simplifying and removing ($\ln$) from both sides, we get

$$(1-V)^3/(1+V)=CX^{-2},\ C=A^{-2}$$

$$-\frac{3}{2}\ln(1-V)+\frac{1}{2}\ln(1+V)=\ln X+\ln A$$
$$\ln(1-V)^{-3/2}+\ln(1+V)^{1/2}=\ln XA$$
$$\ln(1-V)^{-3/2}(1+V)^{1/2}=\ln XA$$
$$(1-V)^{-3/2}(1+V)^{1/2}=XA$$
taking power "−2" on both sides
$$(1-V)^3(1+V)^{-1}=X^{-2}A^{-2}$$
$$put\ V=\frac{Y}{X}$$
$$(1-\frac{Y}{X})^3\left(1+\frac{Y}{X}\right)^{-1}=X^{-2}A^{-2}$$
$$\left(\frac{X-Y}{X}\right)^3\left(\frac{X+Y}{X}\right)^{-1}=X^{-2}A^{-2}$$
$$\frac{(X-Y)^3}{X+Y}X^{-3+1}=X^{-2}A^{-2}$$
$$say, c=A^{-2}$$
$$\frac{(X-Y)^3}{X+Y}=c$$
$$put\ X=x-2,\ Y=y-1$$
$$(x+y-1)^3/x+y-3=c$$

Now substituting $V=\frac{Y}{X}, X=x-2,\ Y=y-1$ and simplifying, we obtain

$$(x-y-1)^3/(x+y-3)=C$$

This is solution of the given differential equation, an implicit one.

Exercise

Solve the following Differential Equations

1. $(x^4+y^4)dx-2x^3ydy=0$

2. $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{x^2}{y^2} + 1$

3. $\left(x^2 e^{\frac{-y}{x}} + y^2\right)dx = xydy$

4. $ydx + \left(y\cos\dfrac{x}{y} - x\right)dy = 0$

5. $\left(x^3 + y^2\sqrt{x^2 + y^2}\right)dx - xy\sqrt{x^2 + y^2}\,dy = 0$

Solve the initial value problems

6. $\left(3x^2 + 9xy + 5y^2\right)dx - \left(6x^2 + 4xy\right)dy = 0, \qquad y(2) = -6$

7. $\left(x + \sqrt{y^2 - xy}\right)\dfrac{dy}{dx} = y, \qquad y\left(\dfrac{1}{2}\right) = 1$

8. $\left(x + ye^{y/x}\right)dx - xe^{y/x}dy = 0, \qquad y(1) = 0$

9. $\dfrac{dy}{dx} - \dfrac{y}{x} = \cosh\dfrac{y}{x}, \qquad y(1) = 0$