MTH 401 – DIFFERENTIAL EQUATIONS

LECTURE 4: EXACT DIFFERENTIAL EQUATIONS

Let us first rewrite the given differential equation

$$\frac{dy}{dx} = f(x,y)$$

into the alternative form

$$M(x,y)dx + N(x,y)dy = 0 \quad \text{where} \quad f(x,y) = -\frac{M(x,y)}{N(x,y)}$$

This equation is an exact differential equation if the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

This condition of exactness insures the existence of a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y), \quad \frac{\partial F}{\partial y} = N(x,y)$$

Method of Solution:

If the given equation is exact then the solution procedure consists of the following steps:

Step 1. Check that the equation is exact by verifying the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Step 2. Write down the system $\frac{\partial F}{\partial x} = M(x,y), \quad \frac{\partial F}{\partial y} = N(x,y)$

Step 3. Integrate either the 1st equation *w. r. to* x or 2nd *w. r. to* y. If we choose the 1st equation then

$$F(x,y) = \int M(x,y)dx + \theta(y)$$

The function $\theta(y)$ is an arbitrary function of y, integration w.r.to x; y being constant.

Step 4. Use second equation in step 2 and the equation in step 3 to find $\theta'(y)$.

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + \theta'(y) = N(x,y)$$

$$\theta'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx$$

Step 5. Integrate to find $\theta(y)$ and write down the function $F(x,y)$;

Step 6. All the solutions are given by the implicit equation

$$F(x,y) = C$$

Step 7. If you are given an IVP, plug in the initial condition to find the constant C.

Caution: x should disappear from $\theta'(y)$. Otherwise something is **wrong!**

Example 1

Solve $(3x^2y+2)dx+(x^3+y)dy=0$

Solution: Here $M=3x^2y+2$ and $N=x^3+y$

$$\frac{\partial M}{\partial y}=3x^2, \frac{\partial N}{\partial x}=3x^2$$

i.e. $$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Hence the equation is exact. The LHS of the equation must be an exact differential i.e. ∃ a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x}=3x^2y+2=M$$

$$\frac{\partial f}{\partial y}=x^3+y=N$$

Integrating 1st of these equations w. r. t. x, have

$$f(x,y)=x^3y+2x+h(y),$$

where $h(y)$ is the constant of integration. Differentiating the above equation w. r. t. y and using 2nd, we obtain

$$\frac{\partial f}{\partial y}=x^3+h'(y)=x^3+y=N$$

Comparing $h'(y)=y$ is independent of x.

or.

Integrating, we have

$$h(y)=\frac{y^2}{2}$$

Thus $$f(x,y)=x^3y+2x+\frac{y^2}{2}$$

Hence the general solution of the given equation is given by

$$f(x,y)=c$$

i.e. $$x^3y+2x+\frac{y^2}{2}=c$$

Note that we could start with the 2nd equation

$$\frac{\partial f}{\partial y}=x^3+y=N$$

to reach on the above solution of the given equation!

Example 2

Solve the initial value problem

$$(2y\sin x\cos x+y^2\sin x)dx+(\sin^2 x-2y\cos x)dy=0.$$

$$y(0)=3.$$

Solution: Here

$$M=2y\sin x\cos x+y^2\sin x$$

and $$N=\sin^2 x-2y\cos x$$

$$\frac{\partial M}{\partial y}=2\sin x\cos x+2y\sin x,$$

$$\frac{\partial N}{\partial x}=2\sin x\cos x+2y\sin x,$$

This implies $$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Thus given equation is exact.

Hence there exists a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x}=2y\sin x\cos x+y^2\sin x=M$$

$$\frac{\partial f}{\partial y}=\sin^2 x-2y\cos x=N$$

Integrating 1st of these w. r. t. x, we have

$$f(x,y) = y\sin^2 x - y^2\cos x + h(y),$$

Differentiating this equation w. r. t. y substituting in $\frac{\partial f}{\partial y} = N$

$$\sin^2 x - 2y\cos x + h'(y) = \sin^2 x - 2y\cos x$$

$$h'(y) = 0 \quad \text{or} \quad h(y) = c_1$$

Hence the general solution of the given equation is

$$f(x,y) = c_2$$

i.e. $$y\sin^2 x - y^2\cos x = C, \text{ where } C = c_1 - c_2$$

Applying the initial condition that when $x = 0, y = 3,$ we have

$$-9 = c$$

since $$y^2\cos x - y\sin^2 x = 9$$

is the required solution.

Example 3:

Solve the DE $$\left(e^{2y} - y\cos xy\right)dx + \left(2xe^{2y} - x\cos x\,y + 2y\right)dy = 0$$

Solution:

The equation is neither separable nor homogenous.

Since, $$\left.\begin{aligned} M(x,y) &= e^{2y} - y\cos xy \\ N(x,y) &= 2xe^{2y} - x\cos xy + 2y \end{aligned}\right\}$$

and

$$\frac{\partial M}{\partial y} = 2e^{2y} + xy\sin xy - \cos xy = \frac{\partial N}{\partial x}$$

Hence the given equation is exact and a function $f(x,y)$ exist for which

$$M(x,y) = \frac{\partial f}{\partial x} \quad \text{and} \quad N(x,y) = \frac{\partial f}{\partial y}$$

which means that

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy \quad \text{and} \quad \frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

Let us start with the second equation i.e.

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

Integrating both sides w.r.to y, we obtain

$$f(x,y) = 2x\int e^{2y}dy - x\int \cos xy dy + 2\int y dy$$

Note that while integrating w.r.to y, x is treated as constant. Therefore

$$f(x,y) = xe^{2y} - \sin xy + y^2 + h(x)$$

h is an arbitrary function of x. From this equation we obtain $\frac{\partial f}{\partial x}$ and equate it to M

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy + h'(x) = e^{2y} - y\cos xy$$

So that $\quad h'(x) = 0 \Rightarrow h(x) = C$

Hence a one-parameter family of solution is given by

$$xe^{2y} - \sin xy + y^2 + c = 0$$

Example 4

Solve $\quad 2xy\,dx + (x^2 - 1)dy = 0$

Solution:

Clearly $\quad M(x,y) = 2xy \quad$ and $\quad N(x,y) = x^2 - 1$

Therefore $\quad \frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$

The equation is exact and $\exists$ a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 - 1$$

We integrate first of these equations to obtain.

$$f(x,y) = x^2 y + g(y)$$

Here $g(y)$ is an arbitrary function y. We find $\frac{\partial f}{\partial y}$ and equate it to $N(x,y)$

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1$$

$$g'(y) = -1 \Rightarrow g(y) = -y$$

Constant of integration need not to be included as the solution is given by

$$f(x, y) = c$$

Hence a one-parameter family of solutions is given by

$$x^2 y - y = c$$

Example 5

Solve the initial value problem

$$(\cos x \sin x - xy^2)dx + y(1 - x^2)dy = 0\,,\; y(0) = 2$$

Solution:

Since
$$\begin{cases} M(x, y) = \cos x . \sin x - x\, y^2 \\ N(x, y) = \quad y(1 - x^2) \end{cases}$$

and
$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}$$

Therefore the equation is exact and $\exists$ a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = \cos x . sin\, x - x\, y^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = y(1 - x^2)$$

Now integrating 2nd of these equations w.r.t. 'y' keeping 'x'constant, we obtain

$$f(x, y) = \frac{y^2}{2}(1 - x^2) + h(x)$$

Differentiate w.r.t. 'x' and equate the result to $M(x, y)$

$$\frac{\partial f}{\partial x} = -xy^2 + h'(x) = \cos x \sin x - xy^2$$

The last equation implies that.

$$h'(x) = \cos x \sin x$$

Integrating w.r.to x, we obtain

$$h(x) = -\int (\cos x)(-\sin x) dx = -\frac{1}{2}\cos^2 x$$

Thus a one parameter family solutions of the given differential equation is

$$\frac{y^2}{2}\left(1-x^2\right)-\frac{1}{2}\cos^2 x = c_1$$

or

$$y^2\left(1-x^2\right)-\cos^2 x = c$$

where $2c_1$ has been replaced by C. The initial condition $y=2$ when $x=0$ demand, that $4(1)-\cos^2(0)=c$ so that $c=3$. Thus the solution of the initial value problem is

$$y^2\left(1-x^2\right)-\cos^2 x = 3$$

Exercise

Determine whether the given equations is exact. If so, please solve.

1. $(\sin y - y\sin x)dx + (\cos x + x\cos y)dy = 0$
2. $\left(1 + \ln x + \dfrac{y}{x}\right)dx = (1 - \ln x)dy$
3. $(y\ln y - e^{-xy})dx + \left(\dfrac{1}{y} + \ln y\right)dy = 0$
4. $\left(2y - \dfrac{1}{x} + \cos 3x\right)\dfrac{dy}{dx} + \dfrac{y}{x^2} - 4x^3 + 3y\sin 3x = 0$
5. $\left(\dfrac{1}{x} + \dfrac{1}{x^2} - \dfrac{y}{x^2 + y^2}\right)dx + \left(ye^y + \dfrac{x}{x^2 + y^2}\right)dy = 0$

Solve the given differential equations subject to indicated initial conditions.

6. $(e^x + y)dx + (2 + x + ye^y)dy = 0, \qquad y(0) = 1$
7. $\left(\dfrac{3y^2 - x^2}{y^5}\right)\dfrac{dy}{dx} + \dfrac{x}{2y^4} = 0, \qquad y(1) = 1$
8. $\left(\dfrac{1}{1+y^2} + \cos x - 2xy\right)\dfrac{dy}{dx} = y(y + \sin x), \qquad y(0) = 1$
9. Find the value of ***k***, so that the given differential equation is exact.
   $(2xy^3 - y\sin xy + ky^4)dx - (20x^3 + x\sin xy)dy = 0$
10. $(6xy^3 + \cos y)dx - (kx^2y^2 - x\sin y)dy = 0$