LECTURE 6: FIRST ORDER LINEAR EQUATIONS

The differential equation of the form:

$$a(x)\frac{dy}{dx} + b(x)y = c(x)$$

is a linear differential equation of first order. The equation can be rewritten in the following **famous form.**

$$\frac{dy}{dx} + p(x)y = q(x)$$

where $p(x)$ and $q(x)$ are continuous functions.

Method of solution:

The general solution of the first order linear differential equation is given by

$$y = \frac{\int u(x)q(x)dx + C}{u(x)}$$

Where $u(x) = \exp\left(\int p(x)dx\right)$

The function $u(x)$ is called the **integrating factor**. If it is an IVP then use it to find the constant C.

Summary:

1. Identify that the equation is 1st order linear equation. Rewrite it in the form

$$\frac{dy}{dx} + p(x)y = q(x)$$

if the equation is not already in this form.

2. Find the integrating factor

$$u(x) = e^{\int p(x)dx}$$

3. Write down the general solution

$$y = \frac{\int u(x)q(x)dx + C}{u(x)}$$

4. If you are given an IVP, use the initial condition to find the constant C.

5. Plug in the calculated value to write the particular solution of the problem.

Example 1:

Solve the initial value problem

$$y' + \tan(x)y = \cos^2(x), \qquad y(0) = 2$$

Solution:

1.The equation is already in the standard form

$$\frac{dy}{dx} + p(x)y = q(x)$$

with

$$\begin{cases} p(x) = \tan x \\ q(x) = \cos^2 x \end{cases}$$

2. Since

$$\int \tan x\, dx = -\ln\cos x = \ln\sec x$$

Therefore, the integrating factor is given by

$$u(x) = e^{\int \tan x\, dx} = \sec x$$

3. Further, because

$$\int \sec x \cos^2 x\, dx = \int \cos x\, dx = \sin x$$

So that the general solution is given by

$$y = \frac{\sin x + C}{\sec x} = (\sin x + C)\cos x$$

4. We use the initial condition $y(0) = 2$ to find the value of the constant C

$$y(0) = C = 2$$

5. Therefore the solution of the initial value problem is

$$y = (\sin x + 2)\cos x$$

Example 2: Solve the IVP $\dfrac{dy}{dt}-\dfrac{2t}{1+t^2}y=\dfrac{2}{1+t^2}, \quad y(0)=0.4$

Solution:

1. The given equation is a 1st order linear and is already in the requisite form

$$\frac{dy}{dx}+p(x)y=q(x)$$

with

$$\begin{cases} p(t)=-\dfrac{2t}{1+t^2} \\ q(t)=\dfrac{2}{1+t^2} \end{cases}$$

2. Since $\displaystyle\int\left(-\frac{2t}{1+t^2}\right)dt=-\ln|1+t^2|$

Therefore, the integrating factor is given by

$$u(t)=e^{\int-\frac{2t}{1+t^2}dt}=(1+t^2)^{-1}$$

3. Hence, the general solution is given by

$$y=\frac{\int u(t)q(t)dt+C}{u(t)}, \quad \int u(t)q(t)dt=\int\frac{2}{(1+t^2)^2}dt$$

Now $\displaystyle\int\frac{2}{(1+t^2)^2}dt=2\int\frac{1+t^2-t^2}{(1+t^2)^2}dt=2\int\left(\frac{1}{1+t^2}-\frac{t^2}{(1+t^2)^2}\right)dt$

The first integral is clearly $\tan^{-1}t$. For the 2nd we will use integration by parts with t as first function and $2t/(1+t^2)^2$ as 2nd function.

$$\int\frac{2t^2}{(1+t^2)^2}dt=t\left(-\frac{1}{1+t^2}\right)+\int\frac{1}{1+t^2}dt=-\frac{t}{1+t^2}+\tan^{-1}(t)$$

$$\int\frac{2}{(1+t^2)^2}dt=2\tan^{-1}(t)+\frac{t}{1+t^2}-\tan^{-1}(t)=\tan^{-1}(t)+\frac{t}{1+t^2}$$

The general solution is: $y=(1+t^2)\left(\tan^{-1}(t)+\dfrac{t}{1+t^2}+C\right)$

4. The condition $y(0)=0.4$ gives $C=0.4$

5. Therefore, solution to the initial value problem can be written as:

$$y=t+(1+t^2)\tan^{-1}(t)+0.4(1+t^2)$$

Example 3:

Find the solution to the problem

$$\cos^2 t \sin t \,.\, y' = -\cos^3 t \,.\, y + 1 \,,\; y\left(\frac{\pi}{4}\right) = 0$$

Solution:

1. The equation is 1st order linear and is not in the standard form

$$\frac{dy}{dx} + p(x)y = q(x)$$

Therefore we rewrite the equation as

$$y' + \frac{\cos\ t}{\sin t} y = \frac{1}{\cos^2 t \sin t}$$

2. Hence, the integrating factor is given by

$$u(t) = e^{\int \frac{\cos t}{\sin t} dt} = e^{\ln|\sin t|} = \sin t$$

3. Therefore, the general solution is given by

$$y = \frac{\int \sin t \frac{1}{\cos^2 t \sin t} dt + C}{\sin t}$$

Since

$$\int \sin t \frac{1}{\cos^2 t\ \sin t} dt = \int \frac{1}{\cos^2 t} dt = \tan t$$

Therefore

$$y = \frac{\tan t + C}{\sin t} = \frac{1}{\cos t} + \frac{C}{\sin t} = \sec t + C \csc t$$

(1) The initial condition $y(\pi/4) = 0$ implies

$$\sqrt{2} + C\sqrt{2} = 0$$

which gives $C = -1$.

(2) Therefore, the particular solution to the initial value problem is

$$y = \sec t - \csc t$$

Example 4

Solve $\left(x+2y^3\right)\frac{dy}{dx}=y$

Solution:

We have

$$\frac{dy}{dx}=\frac{y}{x+2y^3}$$

This equation is not linear in y. Let us regard x as dependent variable and y as independent variable. The equation may be written as

$$\frac{dx}{dy}=\frac{x+2y^3}{y}$$

or

$$\frac{dx}{dy}-\frac{1}{y}x=2y^2$$

Which is linear in x

$$IF=\exp\left[\int\left(-\frac{1}{y}\right)dy\right]=\exp\left[\ln\frac{1}{y}\right]=\frac{1}{y}$$

Multiplying with the $IF=\frac{1}{y}$, we get

$$\frac{1}{y}\frac{dx}{dy}-\frac{1}{y^2}x=2y$$

$$\frac{d}{dy}\left(\frac{x}{y}\right)=2y$$

Integrating, we have

$$\frac{x}{y}=y^2+c$$

$$x=y\left(y^2+c\right)$$

is the required solution.

Example 5

Solve

$$(x-1)^3 \frac{dy}{dx} + 4(x-1)^2 y = x+1$$

Solution:

The equation can be rewritten as

$$\frac{dy}{dx} + \frac{4}{x-1} y = \frac{x+1}{(x-1)^3}$$

Here $$P(x) = \frac{4}{x-1}.$$

Therefore, an integrating factor of the given equation is

$$IF = \exp\left[\int \frac{4dx}{x-1}\right] = \exp\left[\ln(x-1)^4\right] = (x-1)^4$$

Multiplying the given equation by the IF, we get

$$(x-1)^4 \frac{dy}{dx} + 4(x-1)^3 y = x^2 - 1$$

or $$\frac{d}{dx}\left[y(x-1)^4\right] = x^2 - 1$$

Integrating both sides, we obtain

$$y(x-1)^4 = \frac{x^3}{3} - x + c$$

which is the required solution.

Exercise

Solve the following differential equations

1. $\frac{dy}{dx}+\left(\frac{2x+1}{x}\right)y=e^{-2x}$

2. $\frac{dy}{dx}+3y=3x^2e^{-3x}$

3. $x\frac{dy}{dx}+(1+x\cot x)y=x$

4. $(x+1)\frac{dy}{dx}-ny=e^x(x+1)^{n+1}$

5. $(1+x^2)\frac{dy}{dx}+4xy=\frac{1}{(1+x^2)^2}$

6. $\frac{dr}{d\theta}+r\sec\theta=\cos\theta$

7. $\frac{dy}{dx}+y=\frac{1-e^{-2x}}{e^x+e^{-x}}$

8. $dx=(3e^y-2x)dy$

Solve the initial value problems

9. $\frac{dy}{dx}=2y+x(e^{3x}-e^{2x}),\quad y(0)=2$

10. $x(2+x)\frac{dy}{dx}+2(1+x)y=1+3x^2,\quad y(-1)=1$