MTH 401 – DIFFERENTIAL EQUATIONS

LECTURE 7: BERNOULLI EQUATIONS

A differential equation that can be written in the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

is called Bernoulli equation.

Method of solution:

For $n = 0,1$ the equation reduces to 1st order linear DE and can be solved accordingly.

For $n \neq 0,1$ we divide the equation with y^n to write it in the form

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = q(x)$$

and then put

$$v = y^{1-n}$$

Differentiating w.r.t. 'x', we obtain

$$v' = (1-n)y^{-n}y'$$

Therefore the equation becomes

$$\frac{dv}{dx} + (1-n)p(x)v = (1-n)q(x)$$

This is a linear equation satisfied by v. Once it is solved, you will obtain the function

$$y = v^{\frac{1}{(1-n)}}$$

If $n > 1$, then we add the solution $y = 0$ to the solutions found the above technique.

Summary:

1.Identify the equation

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

as Bernoulli equation.

Find $\boldsymbol{n}$. If $n \neq 0,1$ divide by y^n and substitute;

$$v = y^{1-n}$$

2. Through easy differentiation, find the new equation

$$\frac{dv}{dx} + (1-n)p(x)v = (1-n)q(x)$$

3. This is a linear equation. Solve the linear equation to find ***v*****.**

4. Go back to the old function *y* through the substitution $y = v^{1/(1-n)}$.

6. If $n > 1$, then include $y = 0$ to in the solution.

7. If you have an IVP, use the initial condition to find the particular solution.

Example 1: Solve the equation $\frac{dy}{dx} = y + y^3$

Solution:

1. The given differential can be written as

$$\frac{dy}{dx} - y = y^3$$

which is a Bernoulli equation with

$$p(x) = -1, q(x) = 1, \boldsymbol{n=3}.$$

Dividing with y^3 we get

$$y^{-3}\frac{dy}{dx} - y^{-2} = 1$$

Therefore we substitute

$$v = y^{1-3} = y^{-2}$$

2. Differentiating w.r.t. 'x' we have

$$y^{-3}\frac{dy}{dx} = -\frac{1}{2}\left(\frac{dv}{dx}\right)$$

So that the equation reduces to

$$\frac{dv}{dx} + 2v = -2$$

3. This is a linear equation. To solve this we find the integrating factor $u(x)$

$$u(x) = e^{\int 2dx} = e^{2x}$$

The solution of the linear equation is given by

$$v = \frac{\int u(x)q(x)dx + c}{u(x)} = \frac{\int e^{2x}(-2)dx + c}{e^{2x}}$$

Since $$\int e^{2x}(-2)dx = -e^{2x}$$

Therefore, the solution for v is given by

$$v = \frac{-e^{2x} + C}{e^{2x}} = Ce^{-2x} - 1$$

4. To go back to y we substitute $v = y^{-2}$. Therefore the general solution of the given DE is

$$y = \pm\left(Ce^{-2x} - 1\right)^{-\frac{1}{2}}$$

5. Since $n > 1$, we include the $y = 0$ in the solutions. Hence, all solutions are

$$y = 0, \qquad y = \pm(Ce^{-2x} - 1)^{-1/2}$$

Example 2:

Solve $$\frac{dy}{dx} + \frac{1}{x}y = xy^2$$

Solution: In the given equation we identify $P(x) = \frac{1}{x},\ q(x) = x$ and $n = 2$.

Thus the substitution $w = y^{-1}$ gives

$$\frac{dw}{dx} - \frac{1}{x}w = -x.$$

The integrating factor for this linear equation is

$$e^{-\int \frac{dx}{x}} = e^{-\ln|x|} = e^{\ln|x|^{-1}} = x^{-1}$$

Hence $$\frac{d}{dx}\left[x^{-1}w\right] = -1.$$

Integrating this latter form, we get

$$x^{-1}w = -x + c \text{ or } w = -x^2 + cx.$$

Since $w = y^{-1}$, we obtain $y = \dfrac{1}{w}$ or

$$y = \frac{1}{-x^2 + cx}$$

For $n > 0$ the trivial solution $y = 0$ is a solution of the given equation. In this example, $y = 0$ is a singular solution of the given equation.

Example 3:

Solve: $$\frac{dy}{dx} + \frac{xy}{1 - x^2} = xy^{\frac{1}{2}} \qquad (1)$$

Solution: Dividing (1) by $y^{\frac{1}{2}}$, the given equation becomes

$$y^{\frac{-1}{2}}\frac{dy}{dx} + \frac{x}{1 - x^2}y^{\frac{1}{2}} = x \qquad (2)$$

Put $$y^{\frac{1}{2}} = v \text{ or. } \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = \frac{dv}{dx}$$

Then (2) reduces to

$$\frac{dv}{dx} + \frac{x}{2\left(1 - x^2\right)}v = \frac{x}{2} \qquad (3)$$

This is linear in v.

$$\text{I.F} = \exp\left[\int \frac{x}{2\left(1 - x^2\right)}dx\right] = \exp\left[\frac{-1}{4}\ln\left(1 - x^2\right)\right] = \left(1 - x^2\right)^{\frac{-1}{4}}$$

Multiplying (3) by $\left(1 - x^2\right)^{\frac{-1}{4}}$, we get

$$\left(1 - x^2\right)^{\frac{-1}{4}}\frac{dv}{dx} + \frac{x}{2\left(1 - x^2\right)^{5/4}}v = \frac{x}{2\left(1 - x^2\right)^{1/4}}$$

or $$\frac{d}{dx}\left[\left(1 - x^2\right)^{\frac{-1}{4}}v\right] = \frac{-1}{4}\left[-2x\left(1 - x^2\right)^{\frac{-1}{4}}\right]$$

Integrating, we have

$$v\left(1-x^2\right)^{\frac{-1}{4}} = \frac{-1}{4}\frac{\left(1-x^2\right)^{\frac{3}{4}}}{3/4} + c$$

or
$$v = c\left(1-x^2\right)^{1/4} - \frac{1-x^2}{3}$$

or
$$y^{\frac{1}{2}} = c\left(1-x^2\right)^{1/4} - \frac{1-x^2}{3}$$

is the required solution.

Exercise

Solve the following differential equations

1. $x\frac{dy}{dx} + y = y^2 \ln x$

2. $\frac{dy}{dx} + y = xy^3$

3. $\frac{dy}{dx} - y = e^x y^2$

4. $\frac{dy}{dx} = y\left(xy^3 - 1\right)$

5. $x\frac{dy}{dx} - (1+x)y = xy^2$

6. $x^2\frac{dy}{dx} + y^2 = xy$

Solve the initial-value problems

7. $x^2\frac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}$

8. $y^{1/2}\frac{dy}{dx} + y^{3/2} = 1, \quad y(0) = 4$

9. $xy\left(1 + xy^2\right)\frac{dy}{dx} = 1, \quad y(1) = 0$

10. $2\frac{dy}{dx} = \frac{y}{x} - \frac{x}{y^2}, \quad y(1) = 1$