MTH 401 – DIFFERENTIAL EQUATIONS

LECTURE 8: EXAMPLES AND WORKED OUT SOLUTIONS

Example 1: $y' = \dfrac{x^2 + y^2}{xy}$

Solution: $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{xy}$

put $y = wx$ then $\dfrac{dy}{dx} = w + x\dfrac{dw}{dx}$

$$w + x\frac{dw}{dx} = \frac{x^2 + w^2x^2}{x\,x\,w} = \frac{1 + w^2}{w}$$

$$w + x\frac{dw}{dx} = \frac{1}{w} + w$$

$$w\,dw = \frac{dx}{x}$$

Integrating

$$\frac{w^2}{2} = \ln x + \ln c$$

$$\frac{y^2}{2x^2} = \ln|xc|$$

$$y^2 = 2x^2 \ln|xc|$$

Example 2: $\frac{dy}{dx}=\frac{(2\sqrt{xy}-y)}{x}$

Solution: $\frac{dy}{dx}=\frac{(2\sqrt{xy}-y)}{x}$

put $y= wx$

$$w+x\frac{dw}{dx}=\frac{(2\sqrt{xwx}-xw)}{x}$$

$$w+x\frac{dw}{dx}=2\sqrt{w}-w$$

$$x\frac{dw}{dx}=2\sqrt{w}-2w$$

$$\frac{dw}{2(\sqrt{w}-w)}=\frac{dx}{x}$$

$$\int\frac{dw}{2(\sqrt{w}-w)}=\int\frac{dx}{x}$$

$$\int\frac{dw}{2\sqrt{w}(1-\sqrt{w})}=\int\frac{dx}{x}$$

put $\sqrt{w}=t$

We get $\int\frac{1}{1-t}dt=\int\frac{dx}{x}$

$$-\ln|1-t|=\ln|x|+\ln|c|$$

$$-\ln|1-t|=\ln|xc|$$

$$(1-t)^{-1}=xc$$

$$(1-\sqrt{w})^{-1}=xc$$

$$(1-\sqrt{y/x})^{-1}=xc$$

Example 3: $(2y^2x-3)dx+(2yx^2+4)dy=0$

Solution: $(2y^2x-3)dx+(2yx^2+4)dy=0$

Here $M=(2y^2x-3)$ and $N=(2yx^2+4)$

$$\frac{\partial M}{\partial y}=4xy=\frac{\partial N}{\partial x}$$

$$\frac{\partial f}{\partial x}=(2y^2x-3) \quad \text{and} \quad \frac{\partial f}{\partial y}=(2yx^2+4)$$

Integrate w.r.t. 'x'

$f(x,y)=x^2y^2-3x+h(y)$

Differentiate w.r.t. 'y'

$$\frac{\partial f}{\partial y}=2x^2y+h'(y)=2x^2y+4=N$$

$h'(y)=4$

Integrate w.r.t. 'y'

$h(y)=4y+c$

$x^2y^2-3x+4y=C_1$

Example 4: $$\frac{dy}{dx}=\frac{2xye^{(x/y)^2}}{y^2+y^2e^{(x/y)^2}+2x^2e^{(x/y)^2}}$$

Solution: $$\frac{dx}{dy}=\frac{y^2+y^2e^{(x/y)^2}+2x^2e^{(x/y)^2}}{2xye^{(x/y)^2}}$$

put $x/y=w$

Aftersubsitution

$$y\frac{dw}{dy}=\frac{1+e^{w^2}}{2we^{w^2}}$$

$$\frac{dy}{y}=\frac{2we^{w^2}}{1+e^{w^2}}dw$$

Integrating

$\ln|y|=\ln|1+e^{w^2}|+\ln c$

$\ln|y|=\ln|c(1+e^{w^2})|$

$y=c(1+e^{(x/y)^2})$

Example 5: $\frac{dy}{dx}+\frac{y}{x\ln x}=\frac{3x^2}{\ln x}$

Solution: $\frac{dy}{dx}+\frac{y}{x\ln x}=\frac{3x^2}{\ln x}$

$$\frac{dy}{dx}+\frac{1}{x\ln x}y=\frac{3x^2}{\ln x}$$

$$p(x)=\frac{1}{x\ln x} \quad \text{and} \quad q(x)=\frac{3x^2}{\ln x}$$

$$I.F=\exp\left(\int\frac{1}{x\ln x}dx\right)=\ln x$$

Multiply both side by lnx

$$\ln x\frac{dy}{dx}+\frac{1}{x}y=3x^2$$

$$\frac{d}{dx}(y\ln x)=3x^2$$

Integrate

$$y\ln x=\frac{3x^3}{3}+c$$

Example 6: $(y^2e^x+2xy)dx-x^2dy=0$

Solution: Here $M=y^2e^x+2xy \quad N=-x^2$

$$\frac{\partial M}{\partial y}=2ye^x+2x, \qquad \frac{\partial N}{\partial x}=-2x$$

Clearly $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

The given equation is not exact

divide the equation by y^2 to make it exact

$$\left[e^x+\frac{2x}{y}\right]dx+\left[-\frac{x^2}{y^2}\right]dy=0$$

Now $\frac{\partial M}{\partial y}=-\frac{2x}{y^2}=\frac{\partial N}{\partial x}$

Equation is exact

$$\frac{\partial f}{\partial x}=\left[e^x+\frac{2x}{y}\right] \qquad \frac{\partial f}{\partial y}=\left[-\frac{x^2}{y^2}\right]$$

Integrate w.r.t. 'x'

$$f(x,y)=e^x+\frac{x^2}{y}$$

$$e^x+\frac{x^2}{y}=c$$

Example 7:

$$x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1$$

Solution: $x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1$

$$\frac{dy}{dx}+y\left[\frac{x\sin x+\cos x}{x\cos x}\right]=\frac{1}{x\cos x}$$

$$\frac{dy}{dx}+y\left[\tan x+1/x\right]=\frac{1}{x\cos x}$$

$$\text{I.F} = \exp\left(\int(\tan x+1/x)dx\right)=x\sec x$$

$$x\sec x\frac{dy}{dx}+yx\sec x\left[\tan x+1/x\right]=\frac{x\sec x}{x\cos x}$$

$$x\sec x\frac{dy}{dx}+y\left[x\sec x\tan x+\sec x\right]=\sec^2 x$$

$$\frac{d}{dx}\left[xy\sec x\right]=\sec^2 x$$

$$xy\sec x=\tan x+c$$

Example 8: $xe^{2y}\frac{dy}{dx}+e^{2y}=\frac{\ln x}{x}$

Solution: $xe^{2y}\frac{dy}{dx}+e^{2y}=\frac{\ln x}{x}$

put $e^{2y}=u$

$$2e^{2y}\frac{dy}{dx}=\frac{du}{dx}$$

$$\frac{x}{2}\frac{du}{dx}+u=\frac{\ln x}{x}$$

$$\frac{du}{dx}+\frac{2}{x}u=2\frac{\ln x}{x^2}$$

Here $p(x)=2/x$ And $Q(x)=\frac{\ln x}{x^2}$

$$I.F=\exp(\int\frac{2}{x}dx)=x^2$$

$$x^2\frac{du}{dx}+2xu=2\ln x$$

$$\frac{d}{dx}(x^2u)=2\ln x$$

Integrate

$$x^2u=2[x\ln x-x]+c$$

$$x^2e^{2y}=2[x\ln x-x]+c$$

Example 9: $\frac{dy}{dx}+y\ln y=ye^{x}$

Solution: $\frac{dy}{dx}+y\ln y=ye^{x}$

$$\frac{1}{y}\frac{dy}{dx}+\ln y=e^{x}$$

put $\ln y=u$

$$\frac{du}{dx}+u=e^{x}$$

$$I.F.=e^{\int dx}=e^{x}$$

$$\frac{d}{dx}(e^{x}u)=e^{2x}$$

Integrate

$$e^{x}.u=\frac{e^{2x}}{2}+c$$

$$e^{x}\ln y=\frac{e^{2x}}{2}+c$$

Example 10: $2x\csc 2y\frac{dy}{dx}=2x-\ln\tan y$

Solution: $2x\csc 2y\frac{dy}{dx}=2x-\ln\tan y$

put $\ln\tan y=u$

$$\frac{dy}{dx}=\sin y\cos y\frac{du}{dx}$$

$$\frac{2x\sin y\cos y}{2\sin y\cos y}\frac{du}{dx}=2x-u$$

$$x\frac{du}{dx}=2x-u$$

$$\frac{du}{dx}+\frac{1}{x}u=2$$

$$I.F=\exp(\int 1/x\,dx)=x$$

$$x\frac{du}{dx}+u=2x$$

$$\frac{d}{dx}(xu)=2x$$

$$xu=x^2+c$$

$$u=x+cx^{-1}$$

$$\ln\tan y=x+cx^{-1}$$

Example 11: $\frac{dy}{dx}+x+y+1=(x+y)^2e^{3x}$

Solution: $\frac{dy}{dx}+x+y+1=(x+y)^2e^{3x}$

Put $x+y=u$

$$\frac{du}{dx}+u=u^2e^{3x}$$

$$\frac{du}{dx}+u=u^2e^{3x} \text{ (Bernouli's)}$$

$$\frac{1}{u^2}\frac{du}{dx}+\frac{1}{u}=e^{3x}$$

put $1/u=w$

$$-\frac{dw}{dx}+w=e^{3x}$$

$$\frac{dw}{dx}-w=-e^{3x}$$

$$I.F=\exp\left(\int -dx\right)=e^{-x}$$

$$e^{-x}\frac{dw}{dx}-we^{-x}=-e^{2x}$$

$$\frac{d}{dx}(e^{-x}w)=-e^{2x}$$

Integrate

$$e^{-x}w=\frac{-e^{2x}}{2}+c$$

$$\frac{1}{u}=\frac{-e^{3x}}{2}+ce^{x}$$

$$\frac{1}{x+y}=\frac{-e^{3x}}{2}+ce^{x}$$

Example 12: $\frac{dy}{dx}=(4x+y+1)^2$

Solution: $\frac{dy}{dx}=(4x+y+1)^2$

put $4x+y+1=u$

we get

$$\frac{du}{dx}-4=u^2$$

$$\frac{du}{dx}=u^2+4$$

$$\frac{1}{u^2+4}du=dx$$

Integrate

$$\frac{1}{2}\tan^{-1}\frac{u}{2}=x+c$$

$$\tan^{-1}\frac{u}{2}=2x+c_1$$

$$u=2\tan(2x+c_1)$$

$$4x+y+1=2\tan(2x+c_1)$$

$$\text{Example 13}: (x+y)^2 \frac{dy}{dx} = a^2$$

$$\text{Solution}: (x+y)^2 \frac{dy}{dx} = a^2$$

$$\text{put } x+y = u$$

$$u^2 \left(\frac{du}{dx} - 1\right) = a^2$$

$$u^2 \frac{du}{dx} - u^2 = a^2$$

$$\frac{u^2}{u^2+a^2} du = dx$$

Integrate

$$\int \frac{u^2+a^2-a^2}{u^2+a^2} du = \int dx$$

$$\int \left(1 - \frac{a^2}{u^2+a^2}\right) du = \int dx$$

$$u - a\tan^{-1}\frac{u}{a} = x + c$$

$$(x+y) - a\tan^{-1}\frac{x+y}{a} = x + c$$

Example 14: $2y\frac{dy}{dx}+x^2+y^2+x=0$

Solution: $2y\frac{dy}{dx}+x^2+y^2+x=0$

put $x^2+y^2=u$

$$\frac{du}{dx}-2x+u+x=0$$

$$\frac{du}{dx}+u=x$$

$$I.F=Exp(\int dx)=e^x$$

$$e^x\frac{du}{dx}+ue^x=xe^x$$

$$\frac{d}{dx}(e^x u)=xe^x$$

Integrating

$$e^x u=xe^x-e^x+c$$

Example 15: $y' + 1 = e^{-(x+y)} \sin x$

Solution: $y' + 1 = e^{-(x+y)} \sin x$

put $x + y = u$

$$\frac{du}{dx} = e^{-u} \sin x$$

$$\frac{1}{e^{-u}} du = \sin x dx$$

$$e^{u} du = \sin x dx$$

Integrate

$$e^{u} = -\cos x + c$$

$$u = \ln|-\cos x + c|$$

$$x + y = \ln|-\cos x + c|$$

Example 16: $x^4y^2y'+x^3y^3=2x^3-3$

Solution: $x^4y^2y'+x^3y^3=2x^3-3$

put $x^3y^3=u$

$$3x^2y^3+3x^3y^2\frac{dy}{dx}=\frac{du}{dx}$$

$$3x^3y^2\frac{dy}{dx}=\frac{du}{dx}-3x^2y^3$$

$$x^4y^2\frac{dy}{dx}=\frac{x}{3}\frac{du}{dx}-x^3y^3$$

$$\frac{x}{3}\frac{du}{dx}=2x^3-3$$

$$\frac{du}{dx}=6x^2-9/x$$

Integrate

$u=2x^3-9\ln x+c$

$x^3y^3=2x^3-9\ln x+c$

Example 17: $\cos(x+y)dy=dx$

Solution: $\cos(x+y)dy=dx$

put $x+y=v$ or $1+\frac{dy}{dx}=\frac{dv}{dx}$, we get

$$\cos v\left[\frac{dv}{dx}-1\right]=1$$

$$dx=\frac{\cos v}{1+\cos v}dv=\left[1-\frac{1}{1+\cos v}\right]dv$$

$$dx=\left[1-\frac{1}{2}\sec^2\frac{v}{2}\right]dv$$

Integrate

$$x+c=v-\tan\frac{v}{2}$$

$$x+c=v-\tan\frac{x+y}{2}$$