MTH 401 – DIFFERENTIAL EQUATIONS

LECTURE 9: APPLICATION OF FIRST ORDER DIFFERENTIAL EQUATIONS

In order to translate a physical phenomenon in terms of mathematics, we strive for a set of equations that describe the system adequately. This set of equations is called a **Model** for the phenomenon. The basic steps in building such a model consist of the following steps:

Step 1: We clearly state the assumptions on which the model will be based. These assumptions should describe the relationships among the quantities to be studied.

Step 2: Completely describe the parameters and variables to be used in the model.

Step 3: Use the assumptions (from Step 1) to derive mathematical equations relating the parameters and variables (from Step 2).

The mathematical models for physical phenomenon often lead to a differential equation or a set of differential equations. The applications of the differential equations we will discuss in next two lectures include:

- Orthogonal Trajectories.
- Population dynamics.
- Radioactive decay.
- Newton's Law of cooling.
- Carbon dating.
- Chemical reactions.

etc.

Orthogonal Trajectories

- We know that that the solutions of a 1st order differential equation, e.g. separable equations, may be given by an implicit equation

$$F(x, y, C) = 0$$

with 1 parameter C, which represents a family of curves. Member curves can be obtained by fixing the parameter C. Similarly an nth order DE will yields an n-parameter family of curves/solutions.

$$F(x, y, C_1, C_1, \cdots, C_n) = 0$$

- The question arises that whether or not we can turn the problem around: Starting with an n-parameter family of curves, can we find an associated nth order differential equation free of parameters and representing the family. The answer in most cases is yes.
- Let us try to see, with reference to a 1-parameter family of curves, how to proceed if the answer to the question is yes.

1. Differentiate with respect to *x*, and get an equation-involving *x*, *y*, $\frac{dy}{dx}$ and *C*.
2. Using the original equation, we may be able to eliminate the parameter *C* from the new equation.
3. The next step is doing some algebra to rewrite this equation in an explicit form

$$\frac{dy}{dx} = f(x, y)$$

- For illustration we consider an example:

Illustration

Example

Find the differential equation satisfied by the family

$$x^2 + y^2 = C\,x$$

Solution:

1. We differentiate the equation with respect to *x*, to get

$$2x + 2y\frac{dy}{dx} = C$$

2. Since we have from the original equation that

$$C = \frac{x^2 + y^2}{x}$$

then we get

$$2x + 2y\frac{dy}{dx} = \frac{x^2 + y^2}{x}$$

3. The explicit form of the above differential equation is

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

This last equation is the desired DE free of parameters representing the given family.

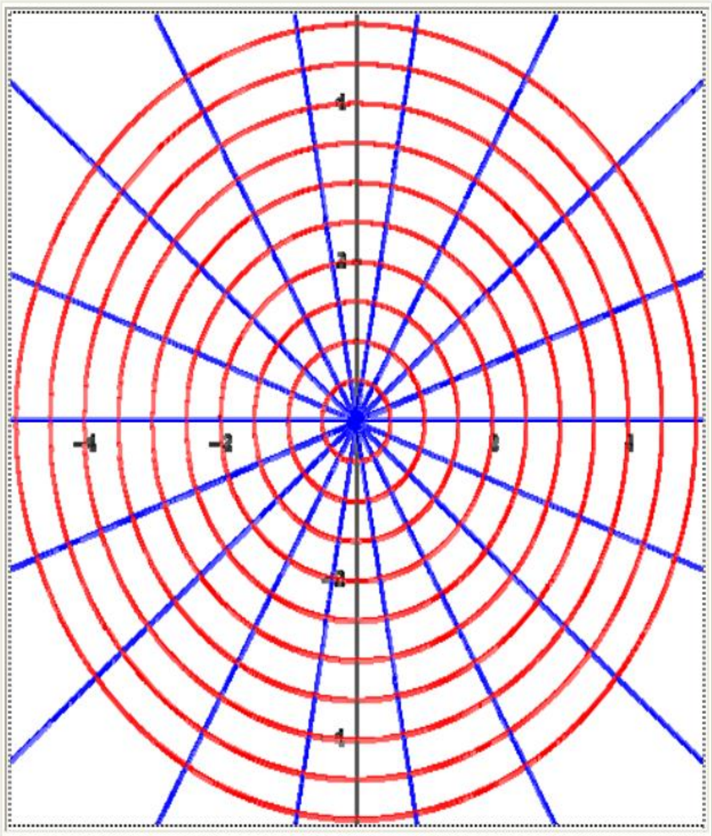
Example.

Let us consider the example of the following two families of curves

$$\begin{cases} y = mx \\ x^2 + y^2 = C^2 \end{cases}$$

The first family describes all the straight lines passing through the origin while the second family describes all the circles centered at the origin

If we draw the two families together on the same graph we get



Clearly whenever one line intersects one circle, the tangent line to the circle (at the point of intersection) and the line are perpendicular i.e. orthogonal to each other. We say that the two families of curves are orthogonal at the point of intersection.

Orthogonal curves:

Any two curves C_1 and C_2 are said to be orthogonal if their tangent lines T_1 and T_2 at their point of intersection are perpendicular. This means that slopes are negative reciprocals of each other, except when T_1 and T_2 are parallel to the coordinate axes.

Orthogonal Trajectories (OT):

When all curves of a family $\Im_1 : G(x, y, c_1) = 0$ orthogonally intersect all curves of another family $\Im_2 : H(x, y, c_2) = 0$ then each curve of the families is said to be orthogonal trajectory of the other.

Example:

As we can see from the previous figure that the family of straight lines $y = mx$ and the family of circles $x^2 + y^2 = C^2$ are orthogonal trajectories.

Orthogonal trajectories occur naturally in many areas of physics, fluid dynamics, in the study of electricity and magnetism etc. For example the lines of force are perpendicular to the equipotential curves i.e. curves of constant potential.

Method of finding Orthogonal Trajectory:

Consider a family of curves $\Im$. Assume that an associated DE may be found, which is given by**:**

$$\frac{dy}{dx} = f(x, y)$$

Since $\frac{dy}{dx}$ gives slope of the tangent to a curve of the family $\Im$ through (x, y).

Therefore, the slope of the line orthogonal to this tangent is $-\frac{1}{f(x, y)}$. So that the slope of the line that is tangent to the orthogonal curve through (x, y) is given by $-\frac{1}{f(x, y)}$. In other words, the family of orthogonal curves are solutions to the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x, y)}$$

The steps can be summarized as follows:

Summary:

In order to find Orthogonal Trajectories of a family of curves $\Im$ we perform the following steps:

Step 1. Consider a family of curves $\Im$ and find the associated differential equation.

Step 2. Rewrite this differential equation in the explicit form

$$\frac{dy}{dx} = f(x, y)$$

Step 3. Write down the differential equation associated to the orthogonal family

$$\frac{dy}{dx} = -\frac{1}{f(x, y)}$$

Step 4. Solve the new equation. The solutions are exactly the family of orthogonal curves.

Step 5. A specific curve from the orthogonal family may be required, something like an IVP.

Example 1

Find the orthogonal Trajectory to the family of circles

$$x^2 + y^2 = C^2$$

Solution:

The given equation represents a family of concentric circles centered at the origin.

Step 1. We differentiate w.r.t. 'x' to find the DE satisfied by the circles.

$$2y\frac{dy}{dx} + 2x = 0$$

Step 2. We rewrite this equation in the explicit form

$$\frac{dy}{dx} = -\frac{x}{y}$$

Step 3. Next we write down the DE for the orthogonal family

$$\frac{dy}{dx} = -\frac{1}{-(x/y)} = \frac{y}{x}$$

Step 4. This is a linear as well as a separable DE. Using the technique of linear equation, we find the integrating factor

$$u(x) = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$$

which gives the solution

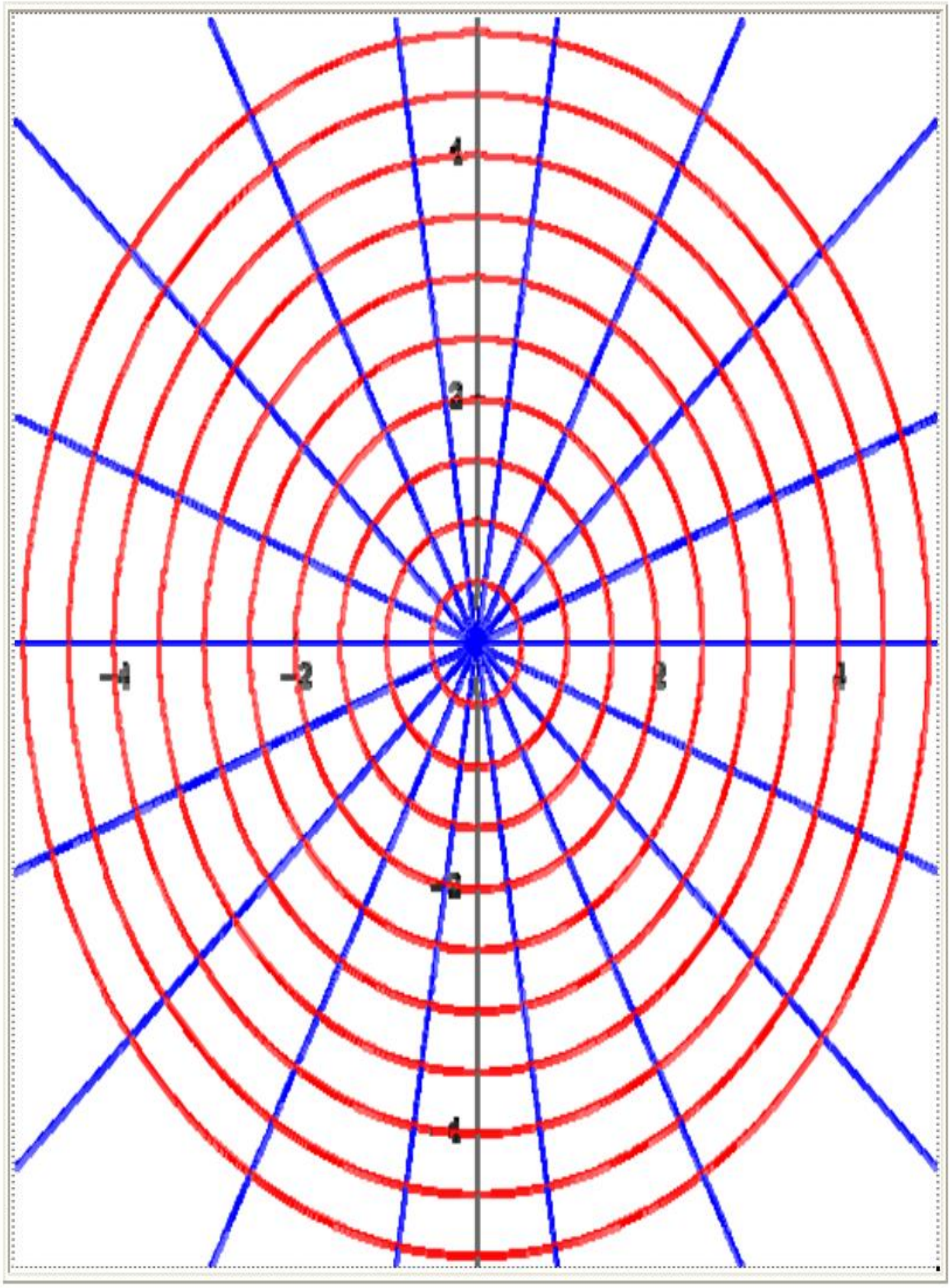
$$y.u(x) = m$$

or

$$y = \frac{m}{u(x)} = mx$$

Which represent a family of straight lines through origin. Hence the family of straight lines $y = mx$ and the family of circles $x^2 + y^2 = C^2$ are Orthogonal Trajectories.

Step 5. A geometrical view of these Orthogonal Trajectories is:



Example 2

Find the Orthogonal Trajectory to the family of circles

$$x^2 + y^2 = 2Cx$$

Solution:

1. We differentiate the given equation to find the DE satisfied by the circles.

$$y\frac{dy}{dx} + x = C, \qquad C = \frac{x^2 + y^2}{2x}$$

2. The explicit differential equation associated to the family of circles is

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

3. Hence the differential equation for the orthogonal family is

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

4. This DE is a homogeneous, to solve this equation we substitute $v = y/x$ or equivalently $y = vx$. Then we have

$$\frac{dy}{dx} = x\frac{dv}{dx} + v \text{ and } \frac{2xy}{x^2 - y^2} = \frac{2v}{1 - v^2}$$

Therefore the homogeneous differential equation in step 3 becomes

$$x\frac{dv}{dx} + v = \frac{2v}{1 - v^2}$$

Algebraic manipulations reduce this equation to the separable form:

$$\frac{dv}{dx} = \frac{1}{x}\left\{\frac{v + v^3}{1 - v^2}\right\}$$

The constant solutions are given by

$$v + v^3 = 0 \Rightarrow v(1 + v^2) = 0$$

The only constant solution is $v = 0$.

To find the non-constant solutions we separate the variables

$$\frac{1 - v^2}{v + v^3}\,dv = \frac{1}{x}\,dx$$

Integrate

$$\int \frac{1-v^2}{v+v^3}\,dv = \int \frac{1}{x}\,dx$$

Resolving into partial fractions the integrand on LHS, we obtain

$$\frac{1-v^2}{v+v^3} = \frac{1-v^2}{v(1+v^2)} = \frac{1}{v} - \frac{2v}{1+v^2}$$

Hence we have

$$\int \frac{1-v^2}{v+v^3}\,dv = \int \left\{\frac{1}{v} - \frac{2v}{1+v^2}\right\} dv = \ln|v| - \ln[v^2+1]$$

Hence the solution of the separable equation becomes

$$\ln|v| - \ln[v^2+1] = \ln|x| + \ln C$$

which is equivalent to

$$\frac{v}{v^2+1} = Cx$$

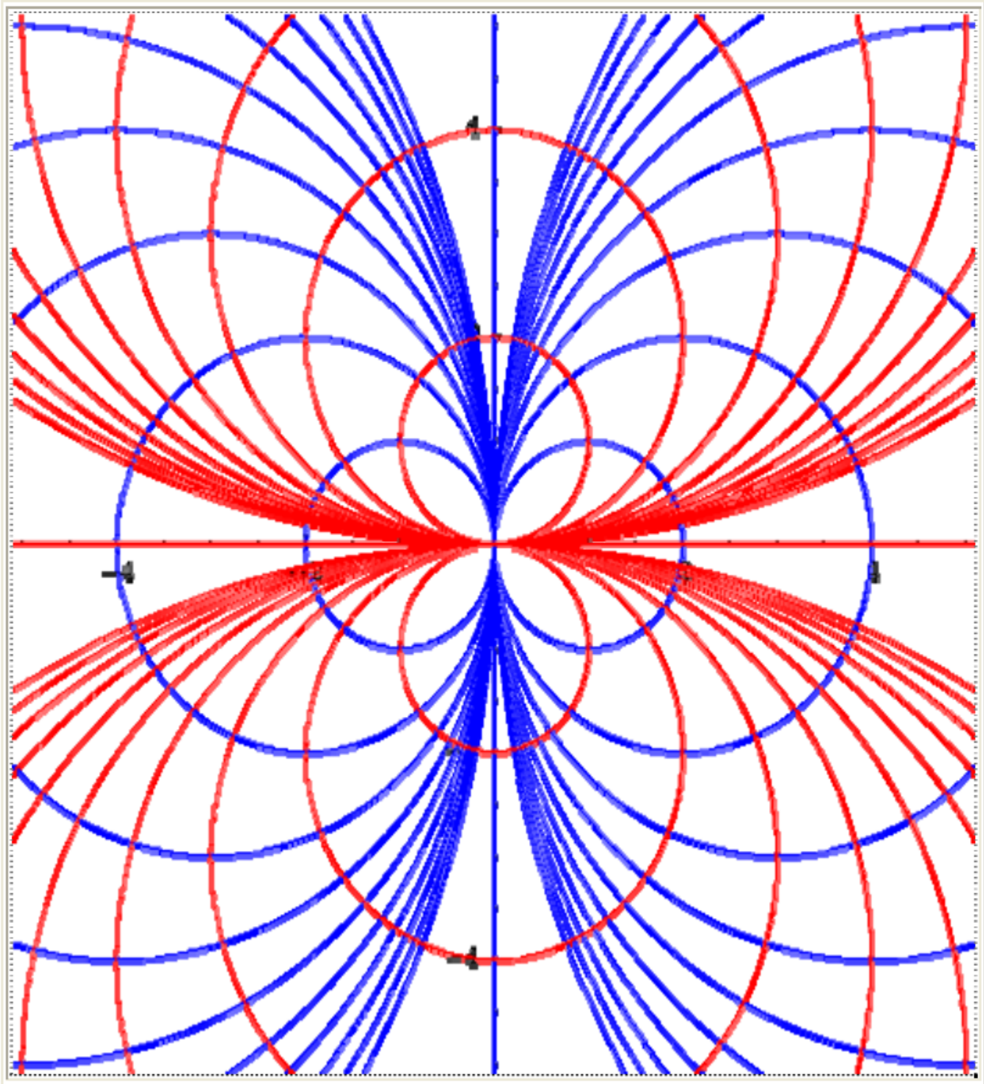
where $C \neq 0$. Hence all the solutions are

$$\begin{cases} v = 0 \\ \dfrac{v}{v^2+1} = Cx \end{cases}$$

We go back to y to get $y = 0$ and $\dfrac{y}{y^2+x^2} = C$ which is equivalent to

$$\begin{cases} y = 0 \\ x^2 + y^2 = my \end{cases}$$

5. Which is x-axis and a family of circles centered on y-axis. A geometrical view of both the families is shown in the next slide.



Population Dynamics

Some natural questions related to population problems are the following:

- What will the population of a certain country after e.g. ten years?
- How are we protecting the resources from extinction?

The easiest population dynamics model is the **exponential model.** This model is based on the assumption:

The rate of change of the population is proportional to the existing population.

If $P(t)$ measures the population of a species at any time t then because of the above mentioned assumption we can write

$$\frac{dP}{dt} = kP$$

where the rate k is constant of proportionality. Clearly the above equation is linear as well as separable. To solve this equation we multiply the equation with the integrating factor e^{-kt} to obtain

$$\frac{d}{dt}\left[P e^{-kt}\right] = 0$$

Integrating both sides we obtain

$$P e^{-kt} = C \quad \text{or} \quad P = C e^{kt}$$

If P_0 is the initial population then $P(0) = P_0$. So that $C = P_0$ and obtain

$$P(t) = P_0 e^{kt}$$

Clearly, we must have $k > 0$ for growth and $k < 0$ for the decay.

Illustration

Example:

The population of a certain community is known to increase at a rate proportional to the number of people present at any time. The population has doubled in 5 years, how long would it take to triple?. If it is known that the population of the community is 10,000 after 3 years. What was the initial population? What will be the population in 30 years?

Solution:

Suppose that P_0 is initial population of the community and $P(t)$ the population at any time t then the population growth is governed by the differential equation

$$\frac{dP}{dt} = kP$$

As we know solution of the differential equation is given by

$$P(t) = P_0 \, e^{kt}$$

Since $P(5) = 2P_0$. Therefore, from the last equation we have

$$2P_0 = P_0 \, e^{5k} \Rightarrow e^{5k} = 2$$

This means that

$$5k = \ln 2 = 0.69315 \quad \text{or} \quad k = \frac{0.69315}{5} = 0.13863$$

Therefore, the solution of the equation becomes

$$P(t) = P_0 \, e^{0.13863\,t}$$

If t_1 is the time taken for the population to triple then

$$3P_0 = P_0 \, e^{0.1386 t_1} \Rightarrow e^{0.1386 t_1} = 3$$

$$t_1 = \frac{\ln 3}{0.1386} = 7.9265 \approx 8 \text{ years}$$

Now using the information $P(3) = 10{,}000$, we obtain from the solution that

$$10{,}000 = P_0 \, e^{(0.13863)(3)} \Rightarrow P_0 = \frac{10{,}000}{e^{0.41589}}$$

Therefore, the initial population of the community was

$$P_0 \approx 6598$$

Hence solution of the model is

$$P(t) = 6598 \, e^{0.13863\,t}$$

So that the population in 30 years is given by

$$P(30) = 6598 e^{(30)(0.13863)} = 6598 e^{4.1589}$$

or $P(30) = (6598)(64.0011)$

or $P(30) \approx 422279$