MTH 401 – DIFFERENTIAL EQUATIONS

LECTURE 10: RADIOACTIVE DECAY

In physics a radioactive substance disintegrates or transmutes into the atoms of another element. Many radioactive materials disintegrate at a rate proportional to the amount present. Therefore, if $A(t)$ is the amount of a radioactive substance present at time t, then the rate of change of $A(t)$ with respect to time t is given by

$$\frac{dA}{dt} = kA$$

where k is a constant of proportionality. Let the initial amount of the material be A_0 then $A(0) = A_0$. As discussed in the population growth model the solution of the differential equation is

$$A(t) = A_0 e^{kt}$$

The constant k can be determined using half-life of the radioactive material.

The half-life of a radioactive substance is the time it takes for one-half of the atoms in an initial amount A_0 to disintegrate or transmute into atoms of another element. The half-life measures stability of a radioactive substance. The longer the half-life of a substance, the more stable it is. If T denotes the half-life then

$$A(T) = \frac{A_0}{2}$$

Therefore, using this condition and the solution of the model we obtain

$$\frac{A_0}{2} = A_0 e^{kt}$$

So that $$kT = -\ln 2$$

Therefore, if we know T, we can get k and vice-versa. The half-life of some important radioactive materials is given in many textbooks of Physics and Chemistry. For example the half-life of $C-14$ is 5568 ± 30 years.

Example 1:

A radioactive isotope has a half-life of 16 days. We have 30 g at the end of 30 days. How much radioisotope was initially present?

Solution: Let $A(t)$ be the amount present at time t and A_0 the initial amount of the isotope. Then we have to solve the initial value problem.

$$\frac{dA}{dt} = kA, \quad A(30) = 30$$

We know that the solution of the IVP is given by

$$A(t) = A_0 e^{kt}$$

If T the half-life then the constant is given k by

$$kT = -\ln 2 \quad \text{or} \quad k = -\frac{\ln 2}{T} = -\frac{\ln 2}{16}$$

Now using the condition $A(30) = 30$, we have

$$30 = A_0 e^{30k}$$

So that the initial amount is given by

$$A_0 = 30e^{-30k} = 30e^{\frac{30\ln 2}{16}} = 110.04\text{ g}$$

Example 2:

A breeder reactor converts the relatively stable uranium 238 into the isotope plutonium 239. After 15 years it is determined that 0.043% of the initial amount A_0 of the plutonium has disintegrated. Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining.

Solution:

Let $A(t)$ denotes the amount remaining at any time t , then we need to find solution to the initial value problem

$$\frac{dA}{dt} = kA, \quad A(0) = A_0$$

which we know is given by

$$A(t) = A_0 e^{kt}$$

If 0.043% disintegration of the atoms of A_0 means that 99.957% of the substance remains. Further 99.957% of A_0 equals $(0.99957)A_0$. So that

$$A(15) = (0.99957)A_0$$

So that

$$A_0 e^{15k} = (0.99957)A_0$$

$$15k = \ln(0.99957)$$

Or $$k = \frac{\ln(0.99957)}{15} = -0.00002867$$

Hence $$A(t) = A_0 e^{-0.00002867\ t}$$

If T denotes the half-life then $A(T) = \frac{A_0}{2}$. Thus

$$\frac{A_0}{2} = A_0 e^{-0.00002867\ T} \quad \text{or} \quad \frac{1}{2} = e^{-0.00002867\ T}$$

$$-0.00002867\ \mathrm{T} = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$T = \frac{\ln 2}{0.00002867} \approx 24{,}180 \text{ years}$$

Newton's Law of Cooling

From experimental observations it is known that the temperature ***T (t)*** of an object changes at a rate proportional to the difference between the temperature in the body and the temperature T_m of the surrounding environment. This is what is known as **Newton's law of cooling**.

If initial temperature of the cooling body is T_0 then we obtain the initial value problem

$$\frac{dT}{dt} = k(T - T_m), \quad T(0) = T_0$$

where k is constant of proportionality. The differential equation in the problem is linear as well as separable.

Separating the variables and integrating we obtain

$$\int \frac{dT}{T - T_m} = \int k\, dt$$

This means that

$$\ln|T - T_m| = kt + C$$

$$T - T_m = e^{kt+C}$$

$$T(t) = T_m + C_1 e^{kt} \quad \text{where} \quad C_1 = e^C$$

Now applying the initial condition $T(0) = T_0$, we see that $C_1 = T_0 - T_m$. Thus the solution of the initial value problem is given by

$$T(t) = T_m + (T_0 - T_m)e^{kt}$$

Hence, If temperatures at times t_1 and t_2 are known then we have

$$T(t_1) - T_m = (T_0 - T_m)e^{kt_1}\ ,\ T(t_2) - T_m = (T_0 - T_m)e^{kt_2}$$

So that we can write

$$\frac{T(t_1) - T_m}{T(t_2) - T_m} = e^{k(t_1 - t_2)}$$

This equation provides the value of k if the interval of time '$t_1 - t_2$' is known and vice-versa.

Example 3:

Suppose that a dead body was discovered at midnight in a room when its temperature was $80°F$. The temperature of the room is kept constant at $60°F$. Two hours later the temperature of the body dropped to $75°\ F$. Find the time of death.

Solution:

Assume that the dead person was not sick, then

$$T(0) = 98.6^o F = T_0 \text{ and } T_m = 60^o F$$

Therefore, we have to solve the initial value problem

$$\frac{dT}{dt} = k(T - 60),\ \ T(0) = 98.6$$

We know that the solution of the initial value problem is

$$T(t) = T_m + (T_0 - T_m)e^{kt}$$

So that

$$\frac{T(t_1) - T_m}{T(t_2) - T_m} = e^{k(t_1 - t_2)}$$

The observed temperatures of the cooling object, i.e. the dead body, are

$$T(t_1) = 80^o F \quad \text{and} \quad T(t_2) = 75^o F$$

Substituting these values we obtain

$$\frac{80 - 60}{75 - 60} = e^{2k} \quad \text{as} \quad t_1 - t_2 = 2 \quad \text{hours}$$

So

$$k = \frac{1}{2}\ln\frac{4}{3} = 0.1438$$

Now suppose that t_1 and t_2 denote the times of death and discovery of the dead body then

$$T(t_1) = T(0) = 98.6^o F \quad \text{and} \quad T(t_2) = 80^o F$$

For the time of death, we need to determine the interval $t_1 - t_2 = t_d$. Now

$$\frac{T(t_1) - T_m}{T(t_2) - T_m} = e^{k(t_1 - t_2)} \Rightarrow \frac{98.6 - 60}{80 - 60} = e^{kt_d}$$

or

$$t_d = \frac{1}{k} \ln \frac{38.6}{20} \approx 4.573$$

Hence the time of death is 7:42 PM.

Carbon Dating

- The isotope *C–14* is produced in the atmosphere by the action of cosmic radiation on nitrogen.
- The ratio of C-14 to ordinary carbon in the atmosphere appears to be constant.
- The proportionate amount of the isotope in all living organisms is same as that in the atmosphere.
- When an organism dies, the absorption of $C-14$ by breathing or eating ceases.
- Thus comparison of the proportionate amount of $C-14$ present, say, in a fossil with constant ratio found in the atmosphere provides a reasonable estimate of its age.
- The method has been used to date wooden furniture in Egyptian tombs.
- Since the method is based on the knowledge of half-life of the radio active $C-14$ (5600 years approximately), the initial value problem discussed in the radioactivity model governs this analysis.

Example:

A fossilized bone is found to contain $1/1000$ of the original amount of *C–14*. Determine the age of the fissile.

Solution:

Let $A(t)$ be the amount present at any time t and A_0 the original amount of C–14. Therefore, the process is governed by the initial value problem.

$$\frac{dA}{dt} = kA, \quad A(0) = A_0$$

We know that the solution of the problem is

$$A(t) = A_0 e^{kt}$$

Since the half life of the carbon isotope is 5600 years. Therefore,

$$A(5600) = \frac{A_0}{2}$$

So that

$$\frac{A_0}{2} = A_0 e^{5600k} \quad \text{or} \quad 5600k = -\ln 2$$

$$k = -0.00012378$$

Hence

$$A(t) = A_0 e^{-(0.00012378)t}$$

If t denotes the time when fossilized bone was found then $A(t) = \frac{A_0}{1000}$

$$\frac{A_0}{1000} = A_0 e^{-(0.00012378)t} \Rightarrow -0.00012378\,t = -\ln 1000$$

Therefore

$$t = \frac{\ln 1000}{0.00012378} = 55{,}800 \text{ years}$$