LECTURE 12: HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

Preliminary theory

- A differential equation of the form

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

or $$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_1(x)y'+a_0(x)y=g(x)$$

where $a_0(x), a_1(x), \dots, a_n(x), g(x)$ are functions of x and $a_n(x) \neq 0$, is called a linear differential equation with variable coefficients.

- However, we shall first study the differential equations with constant coefficients i.e. equations of the type

$$a_n\frac{d^n y}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1\frac{dy}{dx}+a_0y=g(x)$$

where $a_0, a_1, \dots, a_n$ are real constants. This equation is non-homogeneous differential equation and

- If $g(x)=0$ then the differential equation becomes

$$a_n\frac{d^n y}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1\frac{dy}{dx}+a_0y=0$$

which is known as the **associated homogeneous differential equation**.

Initial -Value Problem

For a linear nth-order differential equation, the problem:

Solve: $$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

Subject to: $y(x_0)=y_0, \quad y'(x_0)=y_0', \dots y^{n-1}(x_0)=y_0^{n-1}$

$y_0, y_0', \dots, y_0^{n-1}$ being arbitrary constants, is called an **initial-value problem** (IVP).

The specified values $y(x_0)=y_0$, $y'(x_0)=y_0', \dots, y^{n-1}(x_0)=y_0^{n-1}$ are called initial-conditions.

For $n=2$ the initial-value problem reduces to

Solve: $$a_2(x)\frac{d^2 y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

Subject to: $y(x_0)=y_0, \dots, \; y'(x_0)=y_0'$

Solution of IVP

A function satisfying the differential equation on *I* whose graph passes through (x_0, y_0) such that the slope of the curve at the point is the number y_0' is called solution of the initial value problem.

Theorem: Existence and Uniqueness of Solutions

Let $a_n(x), a_{n-1}(x), ..., a_1(x), a_0(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0,\ \forall\, x \in I$. If $x = x_0 \in I$, then a solution $y(x)$ of the initial-value problem exist on I and is unique.

Example 1

Consider the function $y = 3e^{2x} + e^{-2x} - 3x$

This is a solution to the following initial value problem

$$y'' - 4y = 12x, \quad y(0) = 4,\ y'(0) = 1$$

Since $$\frac{d^2y}{dx^2} = 12e^{2x} + 4e^{-2x}$$

and $$\frac{d^2y}{dx^2} - 4y = 12e^{2x} + 4e^{-2x} - 12e^{2x} - 4e^{-2x} + 12x = 12x$$

Further $y(0) = 3 + 1 - 0 = 4$ and $y'(0) = 6 - 2 - 3 = 1$

Hence $y = 3e^{2x} + e^{-2x} - 3x$

is a solution of the initial value problem.

We observe that

- The equation is linear differential equation.
- The coefficients being constant are continuous.
- The function $g(x) = 12x$ being polynomial is continuous.
- The leading coefficient $a_2(x) = 1 \neq 0$ for all values of x.

Hence the function $y = 3e^{2x} + e^{-2x} - 3x$ is the unique solution.

Example 2

Consider the initial-value problem

$$3y''' + 5y'' - y' + 7y = 0,$$
$$y(1) = 0, \quad y'(1) = 0, \quad y''(1) = 0$$

Clearly the problem possesses the trivial solution $y = 0$.

Since

- The equation is homogeneous linear differential equation.
- The coefficients of the equation are constants.
- Being constant the coefficient are continuous.
- The leading coefficient $a_3 = 3 \neq 0$.

Hence $y = 0$ is the only solution of the initial value problem.

Note: If $a_n = 0$?

If $a_n(x) = 0$ in the differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

for some $x \in I$ then

- Solution of initial-value problem may not be unique.
- Solution of initial-value problem may not even exist.

Example 4
Consider the function

$$y = cx^2 + x + 3$$

and the initial-value problem

$$x^2 y'' - 2xy' + 2y = 6$$
$$y(0) = 3, \qquad y'(0) = 1$$

Then $\qquad y' = 2cx + 1$ and $y'' = 2c$

Therefore $\qquad x^2 y'' - 2xy' + 2y = x^2(2c) - 2x(2cx+1) + 2(cx^2 + x + 3)$

$$= 2cx^2 - 4cx^2 - 2x + 2cx^2 + 2x + 6$$
$$= 6.$$

Also $\qquad y(0) = 3 \Rightarrow c(0) + 0 + 3 = 3$

and $\qquad y'(0) = 1 \Rightarrow 2c(0) + 1 = 1$

So that for any choice of c, the function 'y' satisfies the differential equation and the initial conditions. Hence the solution of the initial value problem is not unique.

Note that

- The equation is linear differential equation.
- The coefficients being polynomials are continuous everywhere.
- The function $g(x)$ being constant is constant everywhere.
- The leading coefficient $a_2(x) = x^2 = 0$ at $x = 0 \in (-\infty, \infty)$.

Hence $a_2(x) = 0$ brought non-uniqueness in the solution

Boundary-value problem (BVP)

For a 2nd order linear differential equation, the problem

Solve: $a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$

Subject to: $y(a)=y_0,\quad y(b)=y_1$

is called a ***boundary-value problem***. The specified values $y(a)=y_0$, and $y(b)=y_1$ are called ***boundary conditions.***

Solution of BVP

A solution of the boundary value problem is a function satisfying the differential equation on some interval I, containing a and b, whose graph passes through two points (a, y_0) and (b, y_1).

Example 5

Consider the function

$$y=3x^2-6x+3$$

We can prove that this function is a solution of the boundary-value problem

$$x^2y''-2xy'+2y=6,$$
$$y(1)=0,\qquad y(2)=3$$

Since $\frac{dy}{dx}=6x-6,\ \frac{d^2y}{dx^2}=6$

Therefore $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=6x^2-12x^2+12x+6x^2-12x+6=6$

Also $y(1)=3-6+3=0,\quad y(2)=12-12+3=3$

Therefore, the function 'y' satisfies both the differential equation and the boundary conditions. Hence y is a solution of the boundary value problem.

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Possible Boundary Conditions

For a 2nd order linear non-homogeneous differential equation

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

all the possible pairs of boundary conditions are

$$y(a)=y_0,\qquad y(b)=y_1,$$
$$y'(a)=y'_0,\qquad y(b)=y_1,$$
$$y(a)=y_0,\qquad y'(b)=y'_1,$$
$$y'(a)=y'_0,\qquad y'(b)=y'_1$$

where y_0, y'_0, y_1 and y'_1 denote the arbitrary constants.

In General

All the four pairs of conditions mentioned above are just special cases of the general boundary conditions

$$\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1$$
$$\alpha_2 y(b) + \beta_2 y'(b) = \gamma_2$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \{0,1\}$

Note that

A boundary value problem may have

- Several solutions.
- A unique solution, or
- No solution at all.

Example 1

Consider the function

$$y = c_1 \cos 4x + c_2 \sin 4x$$

and the boundary value problem

$$y'' + 16y = 0, \quad y(0) = 0, \quad y(\pi/2) = 0$$

Then

$$y' = -4c_1 \sin 4x + 4c_2 \cos 4x$$
$$y'' = -16(c_1 \cos 4x + c_2 \sin 4x)$$
$$y'' = -16y$$
$$y'' + 16y = 0$$

Therefore, the function

$$y = c_1 \cos 4x + c_2 \sin 4x$$

satisfies the differential equation

$$y'' + 16y = 0.$$

Now apply the boundary conditions

Applying $y(0) = 0$

We obtain

$$0 = c_1 \cos 0 + c_2 \sin 0$$
$$\Rightarrow c_1 = 0$$

So that

$$y = c_2 \sin 4x.$$

But when we apply the 2nd condition $y(\pi/2) = 0$, we have

$$0 = c_2 \sin 2\pi$$

Since $\sin 2\pi = 0$, the condition is satisfied for any choice of c_2, solution of the problem is the one-parameter family of functions

$$y = c_2 \sin 4x$$

Hence, there are an ***infinite number of solutions*** of the boundary value problem.

Example 2

Solve the boundary value problem

$$y'' + 16y = 0$$

$$y(0) = 0, \; y\left(\frac{\pi}{8}\right) = 0,$$

Solution:

As verified in the previous example that the function

$$y = c_1 \cos 4x + c_2 \sin 4x$$

satisfies the differential equation

$$y'' + 16y = 0$$

We now apply the boundary conditions

$$y(0) = 0 \Rightarrow 0 = c_1 + 0$$

and $\quad y(\pi/8) = 0 \Rightarrow 0 = 0 + c_2$

So that $\quad c_1 = 0 = c_2$

Hence

$$y = 0$$

is the only solution of the boundary-value problem.

Example 3

Solve the differential equation

$$y'' + 16y = 0$$

subject to the boundary conditions

$$y(0) = 0, \quad y(\pi/2) = 1$$

Solution:

As verified in an earlier example that the function

$$y = c_1 \cos 4x + c_2 \sin 4x$$

satisfies the differential equation

$$y'' + 16y = 0$$

We now apply the boundary conditions

$$y(0) = 0 \Rightarrow 0 = c_1 + 0$$

Therefore $\quad c_1 = 0$

So that $\quad y = c_2 \sin 4x$

However $\quad y(\pi/2) = 1 \Rightarrow c_2 \sin 2\pi = 1$

or $\quad 1 = c_2.0 \Rightarrow 1 = 0$

This is a clear contradiction. Therefore, the boundary value problem has ***no solution***.

Definition: Linear Dependence

A set of functions

$$\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

is said to be ***linearly dependent*** on an interval I if $\exists$ constants $c_1, c_2, \ldots, c_n$ not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots. + c_n f_n(x) = 0, \quad \forall\, x \in I$$

Definition: Linear Independence

A set of functions

$$\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

is said to be linearly independent on an interval I if

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0, \quad \forall\, x \in I,$$

only when

$$c_1 = c_2 = \cdots = c_n = 0.$$

Case of two functions:

If $n = 2$ then the set of functions becomes

$$\{f_1(x), f_2(x)\}$$

If we suppose that

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

Also that the functions are linearly dependent on an interval I then either $c_1 \neq 0$ or $c_2 \neq 0$.

Let us assume that $c_1 \neq 0$, then

$$f_1(x) = -\frac{c_2}{c_1} f_2(x);$$

Hence $f_1(x)$ is the constant multiple of $f_2(x)$.

Conversely, if we suppose

$$f_1(x) = c_2\, f_2(x)$$

Then $\quad (-1) f_1(x) + c_2 f_2(x) = 0,\ \forall x \in I$

So that the functions are linearly dependent because $c_1 = -1$.

Hence, we conclude that:

- Any two functions $f_1(x)$ and $f_2(x)$ are linearly dependent on an interval I if and only if one is the constant multiple of the other.
- Any two functions are linearly independent when neither is a constant multiple of the other on an interval I.
- In general a set of n functions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ is linearly dependent if at least one of them can be expressed as a linear combination of the remaining.

Example 1

The functions

$$f_1(x) = \sin 2x, \qquad \forall x \in (-\infty, \infty)$$

$$f_2(x) = \sin x \cos x, \quad \forall x \in (-\infty, \infty)$$

If we choose $c_1 = \frac{1}{2}$ and $c_2 = -1$ then

$$c_1 \sin 2x + c_2 \sin x \cos x = \frac{1}{2}(2 \sin x \cos x) - \sin x \cos x = 0$$

Hence, the two functions $f_1(x)$ and $f_2(x)$ are linearly dependent.

Example 3

Consider the functions

$$f_1(x) = \cos^2 x, \ f_2(x) = \sin^2 x, \quad \forall x \in (-\pi/2, \pi/2),$$

$$f_3(x) = \sec^2 x, \ f_4(x) = \tan^2 x, \quad \forall x \in (-\pi/2, \pi/2)$$

If we choose $c_1 = c_2 = 1, c_3 = -1, c_4 = 1$, then

$$\begin{aligned} & c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + c_4 f_4(x) \\ &= c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x \\ &= \cos^2 x + \sin^2 x + -1 - \tan^2 x + \tan^2 x \\ &= 1 - 1 + 0 = 0 \end{aligned}$$

Therefore, the given functions are linearly dependent.

Note that

The function $f_3(x)$ can be written as a linear combination of other three functions $f_1(x), f_2(x)$ and $f_4(x)$ because $\sec^2 x = \cos^2 x + \sin^2 x + \tan^2 x$.

Example 3

Consider the functions

$$f_1(x) = 1 + x, \ \forall x \in (-\infty, \infty)$$
$$f_2(x) = x, \quad \forall x \in (-\infty, \infty)$$
$$f_3(x) = x^2, \quad \forall x \in (-\infty, \infty)$$

Then

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

means that

$$c_1(1 + x) + c_2 x + c_3 x^2 \qquad = 0$$

or $$c_1 + (c_1 + c_2)x + c_3 x^2 \qquad = 0$$

Equating coefficients of x and x^2 constant terms we obtain

$$c_1 = 0 = c_3$$
$$c_1 + c_2 = 0$$

Therefore $c_1 = c_2 = c_3 = 0$

Hence, the three functions $f_1(x), f_2(x)$ and $f_3(x)$ are linearly independent.

Definition: Wronskian

Suppose that the function $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least $n-1$ derivatives then the determinant

$$\begin{vmatrix} f_1 & f_2 & \cdots\cdots & f_n \\ f_1^{/} & f_2^{/} & \cdots\cdots & f_n^{/} \\ \vdots & \vdots & & \vdots \\ f_1^{n-1} & f_2^{n-1} & \cdots\cdots & f_n^{n-1} \end{vmatrix}$$

is called Wronskian of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ and is denoted by $W(f_1(x), f_1(x), \ldots, f_1(x))$.

Theorem: Criterion for Linearly Independent Functions

Suppose the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possess at least *n-1* derivatives on an interval I. If

$$W(f_1(x), f_2(x), \ldots, f_n(x)) \neq 0$$

for at least one point in I, then functions $f_1(x), f_2(x), \ldots, f_n(x)$ are linearly independent on the interval I.

Note that

This is only a sufficient condition for linear independence of a set of functions.

In other words

If $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least $n-1$ derivatives on an interval and are linearly dependent on I, then

$$W(f_1(x), f_2(x), \ldots, f_n(x)) = 0, \quad \forall x \in I$$

However, the converse is not true. i.e. a Vanishing Wronskian does not guarantee linear dependence of functions.

Example 1

The functions

$$f_1(x) = \sin^2 x$$
$$f_2(x) = 1 - \cos 2x$$

are linearly dependent because

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

We observe that for all $x \in (-\infty, \infty)$

$$W(f_1(x), f_2(x)) = \begin{vmatrix} \sin^2 x & 1 - \cos 2x \\ 2\sin x \cos x & 2\sin 2x \end{vmatrix}$$

$$= 2\sin^2 x \sin 2x - 2\sin x \cos x$$
$$+ 2\sin x \cos x \cos 2x$$

$$= \sin 2x[2\sin^2 x - 1 + \cos 2x]$$
$$= \sin 2x[2\sin^2 x - 1 + \cos^2 x - \sin^2 x]$$
$$= \sin 2x[\sin^2 x + \cos^2 x - 1]$$
$$= 0$$

Example 2

Consider the functions

$$f_1(x) = e^{m_1 x}, f_2(x) = e^{m_2 x}, \quad m_1 \neq m_2$$

The functions are linearly independent because

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

if and only if $\quad c_1 = 0 = c_2 \quad$ as $\quad m_1 \neq m_2$

Now for all $x \in R$

$$W\left(e^{m_1 x}, e^{m_2 x}\right) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix}$$

$$= (m_2 - m_1) e^{(m_1 + m_2)x}$$

$$\neq 0$$

Thus f_1 and f_2 are linearly independent of any interval on x-axis.

Example 3

If α and β are real numbers, $\beta \neq 0$, then the functions

$$y_1 = e^{\alpha x} \cos \beta x \text{ and } y_2 = e^{\alpha x} \sin \beta x$$

are linearly independent on any interval of the x-axis because

$$W\left(e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x\right)$$

$$= \begin{vmatrix} e^{\alpha x} \cos \beta x & e^{\alpha x} \sin \beta x \\ -\beta e^{\alpha x} \sin \beta x + \alpha e^{\alpha x} \cos \beta x & \beta e^{\alpha x} \cos \beta x + \alpha e^{\alpha x} \sin \beta x \end{vmatrix}$$

$$= \beta e^{2\alpha x}\left(\cos^2 \beta x + \sin^2 \beta x\right) = \beta e^{2\alpha x} \neq 0.$$

Example 4

The functions

$$f_1(x) = e^x,\ f_2(x) = xe^x, \text{ and } f_3(x) = x^2 e^x$$

are linearly independent on any interval of the x-axis because for all $x \in R$, we have

$$W\left(e^x, xe^x, x^2 e^x\right) = \begin{vmatrix} e^x & xe^x & x^2 e^x \\ e^x & xe^x + e^x & x^2 e^x + 2xe^x \\ e^x & xe^x + 2e^x & x^2 e^x + 4xe^x + 2e^x \end{vmatrix}$$

$$= 2e^{3x} \neq 0$$

Exercise

1. Given that
$$y = c_1 e^{x} + c_2 e^{-x}$$
is a two-parameter family of solutions of the differential equation
$$y'' - y = 0$$
on $(-\infty, \infty)$, find a member of the family satisfying the boundary conditions
$$y(0) = 0, \quad y'(1) = 1.$$
2. Given that
$$y = c_1 + c_2 \cos x + c_3 \sin x$$
is a three-parameter family of solutions of the differential equation
$$y''' + y' = 0$$
on the interval $(-\infty, \infty)$, find a member of the family satisfying the initial conditions $y(\pi) = 0, \ y'(\pi) = 2, y''(\pi) = -1$.
3. Given that
$$y = c_1 x + c_2 x \ln x$$
is a two-parameter family of solutions of the differential equation $x^2 y'' - xy' + y = 0$ on $(-\infty, \infty)$. Find a member of the family satisfying the initial conditions
$$y(1) = 3, \quad y'(1) = -1.$$

Determine whether the functions in problems 4-7 are linearly independent or dependent on $(-\infty, \infty)$.

4. $f_1(x) = x, \quad f_2(x) = x^2, \quad f_3(x) = 4x - 3x^2$
5. $f_1(x) = 0, \quad f_2(x) = x, \ f_3(x) = e^{x}$
6. $f_1(x) = \cos 2x, \quad f_2(x) = 1, \quad f_3(x) = \cos^2 x$
7. $f_1(x) = e^{x}, \quad f_2(x) = e^{-x}, \quad f_3(x) = \sinh x$

Show by computing the Wronskian that the given functions are linearly independent on the indicated interval.

8. $\tan x, \ \cot x; \quad (-\infty, \infty)$
9. $e^{x}, e^{-x}, e^{4x}; \quad (-\infty, \infty)$
10. $x, x \ln x, x^2 \ln x; \quad (0, \infty)$