## LECTURE 13: SOLUTIONS OF HIGHER ORDER LINEAR EQUATIONS

### Preliminary Theory

- In order to solve an $n$th order non-homogeneous linear differential equation
$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$
we first solve the associated homogeneous differential equation
$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

- Therefore, we first concentrate upon the preliminary theory and the methods of solving the homogeneous linear differential equation.

- We recall that a function $y=f(x)$ that satisfies the associated homogeneous equation
$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$
is called solution of the differential equation.

### Superposition Principle

Suppose that $y_1, y_2, \ldots, y_n$ are solutions on an interval $I$ of the homogeneous linear differential equation

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

Then

$$y=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x),$$

$c_1, c_2, \ldots, c_n$ being arbitrary constants is also a solution of the differential equation.

### Note that

- A constant multiple $y=c_1y_1(x)$ of a solution $y_1(x)$ of the homogeneous linear differential equation is also a solution of the equation.
- The homogeneous linear differential equations always possess the trivial solution $y=0$.

- The superposition principle is a property of linear differential equations and it does not hold in case of non-linear differential equations.

**Example 1**

The functions

$$y_1 = e^x, y_2 = c^{2x}, \text{ and } y_3 = e^{3x}$$

all satisfy the homogeneous differential equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$$

on $(-\infty, \infty)$. Thus $y_1, y_2$ and $y_3$ are all solutions of the differential equation

Now suppose that

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x}.$$

Then

$$\frac{dy}{dx} = c_1e^x + 2c_2e^{2x} + 3c_3e^{3x}.$$

$$\frac{d^2y}{dx^2} = c_1e^x + 4c_2e^{2x} + 9c_3e^{3x}.$$

$$\frac{d^3y}{dx^3} = c_1e^x + 8c_2e^{2x} + 27c_3e^{3x}.$$

Therefore

$$\begin{aligned}&\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y \\ &= c_1\left(e^x - 6e^x + 11e^x - 6e^x\right) + c_2\left(8e^{2x} - 24e^{2x} + 22e^{2x} - 6e^{2x}\right) \\ &\quad + c_3\left(27e^{3x} - 54e^{3x} + 33e^{3x} - 6e^{3x}\right) \\ &= c_1(12-12)e^x + c_2(30-30)e^{2x} + c_3(60-60)e^{3x} \\ &= 0\end{aligned}$$

Thus

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x}.$$

is also a solution of the differential equation.

**Example 2**

The function

$$y = x^2$$

is a solution of the homogeneous linear equation

$$x^2 y'' - 3xy' + 4y = 0$$

on $(0,\infty)$.

Now consider

$$y = cx^2$$

Then $\quad y' = 2cx \quad \text{and} \quad y'' = 2c$

So that $\quad x^2 y'' - 3xy' + 4y = 2cx^2 - 6cx^2 + 4cx^2 = 0$

Hence the function

$$y = cx^2$$

is also a solution of the given differential equation.

**The Wronskian**

Suppose that $y_1, y_2$ are 2 solutions, on an interval $I$, of the second order homogeneous linear differential equation

$$a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

Then either $\quad W(y_1, y_2) = 0, \quad \forall x \in I$

or $\quad W(y_1, y_2) \neq 0, \quad \forall x \in I$

To verify this we write the equation as

$$\frac{d^2 y}{dx^2} + \frac{Pdy}{dx} + Qy = 0$$

Now $\quad W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$

Differentiating $w.r.to\, x$, we have

$$\frac{dW}{dx} = y_1 y_2'' - y_1'' y_2$$

**Since** $y_1$ and $y_2$ are solutions of the differential equation

$$\frac{d^2y}{dx^2}+\frac{Pdy}{dx}+Qy=0$$

Therefore

$$y_1''+Py_1'+Qy_1=0$$

$$y_2''+Py_2'+Qy_2=0$$

Multiplying 1st equation by $y_2$ and 2nd by $y_1$ the have

$$y_1''y_2+Py_1'y_2+Qy_1y_2=0$$

$$y_1y_2''+Py_1y_2'+Qy_1y_2=0$$

Subtracting the two equations we have:

$$(y_1y_2''-y_2y_1'')+P(y_1y_2'-y_1'y_2)=0$$

or $$\frac{dW}{dx}+PW=0$$

This is a linear 1st order differential equation in $W$, whose solution is

$$W=ce^{-\int Pdx}$$

Therefore

- If $c\neq 0$ then $W(y_1,y_2)\neq 0,\quad \forall\, x\in I$
- If $c=0$ then $W(y_1,y_2)=0,\quad \forall\, x\in I$

Hence Wronskian of $y_1$ and $y_2$ is either identically zero or is never zero on $I$.

**In general**

If $y_1,y_2,\ldots,y_n$ are $n$ solutions, on an interval $I$, of the homogeneous $n$th order linear differential equation with constants coefficients

$$a_n\frac{d^ny}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1\frac{dy}{dx}+a_0y=0$$

Then

Either $$W(y_1,y_2,\ldots,y_n)=0,\ \forall x\in I$$

or $$W(y_1,y_2,\ldots,y_n)\neq 0,\ \forall x\in I$$

**Linear Independence of Solutions:**

Suppose that

$$y_1, y_2, \ldots, y_n$$

are $n$ solutions, on an interval $I$ , of the homogeneous linear $n$th-order differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Then the set of solutions is linearly independent on $I$ if and only if

$$W(y_{1,} y_2, \ldots, y_n) \neq 0$$

**In other words**

The solutions

$$y_1, y_2, \ldots, y_n$$

are linearly dependent if and only if

$$W(y_{1,} y_2, \ldots, y_n) = 0, \quad \forall x \in I$$

**Fundamental Set of Solutions**

A set

$$\{y_1, y_2, \ldots, y_n\}$$

of $n$ linearly independent solutions, on interval $I$ , of the homogeneous linear $n$th-order differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

is said to be a fundamental set of solutions on the interval $I$ .

**Existence of a Fundamental Set**

There always exists a fundamental set of solutions for a linear $n$th-order homogeneous differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

on an interval $I$.

**General Solution-Homogeneous Equations**

Suppose that

$$\{y_1, y_2, \ldots, y_n\}$$

is a fundamental set of solutions, on an interval *I*, of the homogeneous linear *n*th-order differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Then the general solution of the equation on the interval *I* is defined to be

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

Here $c_1, c_2, \ldots, c_n$ are arbitrary constants.

**Example 1**

The functions

$$y_1 = e^{3x} \text{ and } y_2 = e^{-3x}$$

are solutions of the differential equation

$$y'' - 9y = 0$$

Since $\quad W\left(e^{3x}, e^{-3x}\right) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0, \quad \forall x \in I$

Therefore $y_1$ and $y_2$ from a fundamental set of solutions on $(-\infty, \infty)$. **Hence** general solution of the differential equation on the $(-\infty, \infty)$ is

$$y = c_1 e^{3x} + c_2 e^{-3x}$$

**Example 2**

Consider the function $y = 4\sinh 3x - 5e^{-3x}$

Then $\quad y' = 12\cosh 3x + 15e^{-3x}, \qquad y'' = 36\sinh 3x - 45e^{-3x}$

$\Rightarrow \quad y'' = 9\left(4\sinh 3x - 5e^{-3x}\right) \quad$ or $\quad y'' = 9y$,

Therefore $\quad y'' - 9y = 0$

Hence $$y = 4\sinh 3x - 5e^{-3x}$$

is a particular solution of differential equation.

$$y'' - 9y = 0$$

The general solution of the differential equation is

$$y = c_1 e^{3x} + c_2 e^{-3x}$$

Choosing $c_1 = 2, c_2 = -7$

We obtain

$$y = 2e^{3x} - 7e^{-3x}$$

$$y = 2e^{3x} - 2e^{-3x} - 5e^{-3x}$$

$$y = 4\left(\frac{e^{3x} - e^{-3x}}{2}\right) - 5e^{-3x}$$

$$y = 4\sinh 3x - 5e^{-3x}$$

Hence, the particular solution has been obtained from the general solution.

**Example 3**

Consider the differential equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$$

and suppose that $y_1 = e^x,\ y_2 = e^{2x}$ and $y_3 = e^{3x}$

Then

$$\frac{dy_1}{dx} = e^x = \frac{d^2y_1}{dx^2} = \frac{d^3y_1}{dx^3}$$

Therefore

$$\frac{d^3y_1}{dx^3} - 6\frac{d^2y_1}{dx^2} + 11\frac{dy_1}{dx} - 6y_1 = e^x - 6e^x + 11e^x - 6e^x$$

or

$$\frac{d^3y_1}{dx^3} - 6\frac{d^2y_1}{dx^2} + 11\frac{dy_1}{dx} - 6y_1 = 12e^x - 12e^x = 0$$

Thus the function $y_1$ is a solution of the differential equation. Similarly, we can verify that the other two functions i.e. $y_2$ and $y_3$ also satisfy the differential equation.

Now for all $x \in R$

$$W\left(e^x, e^{2x}, e^{3x}\right) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0 \quad \forall\, x \in I$$

Therefore $y_1, y_2,$ and $y_3$ form a fundamental solution of the differential equation on $(-\infty, \infty)$. We conclude that

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

is the general solution of the differential equation on the interval $(-\infty,\infty)$.

**Non-Homogeneous Equations**

A function $y_p$ that satisfies the non-homogeneous differential equation

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

and is free of parameters is called the particular solution of the differential equation

**Example 1**

Suppose that

$$y_p=3$$

Then $$y_p''=0$$

So that

$$y_p''+9y_p=0+9(3)$$
$$=27$$

Therefore

$$y_p=3$$

is a particular solution of the differential equation

$$y_p''+9y_p=27$$

**Example 2**

Suppose that

$$y_p=x^3-x$$

Then $$y_p'=3x^2-1,\quad y_p''=6x$$

Therefore $$x^2y_p''+2xy_p'-8y_p=x^2(6x)+2x\left(3x^2-1\right)-8\left(x^3-x\right)$$
$$=4x^3+6x$$

Therefore

$$y_p=x^3-x$$

is a particular solution of the differential equation

$$x^2y''+2xy'-8y=4x^3+6x$$

**Complementary Function**

The general solution

$$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

of the homogeneous linear differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

is known as the complementary function for the non-homogeneous linear differential equation.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

**General Solution of Non-Homogeneous Equations**

Suppose that

- The particular solution of the non-homogeneous equation

  $$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

  is $y_p$.

- The complementary function of the non-homogeneous differential equation

  $$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

  is

  $$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n .$$

- Then general solution of the non-homogeneous equation on the interval $I$ is given by

  $$y = y_c + y_p$$

  or

  $$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x) = y_c(x) + y_p(x)$$

**Hence**

*General Solution = Complementary solution + any particular solution.*

**Example**

Suppose that

$$y_p = -\frac{11}{12} - \frac{1}{2}x$$

Then $$y'_p = -\frac{1}{2},\ y''_p = 0 = y'''_p$$

$\therefore$ $$\frac{d^3 y_p}{dx^3} - 6\frac{d^2 y_p}{dx^2} + 11\frac{dy_p}{dx} - 6y_p = 0 - 0 - \frac{11}{2} + \frac{11}{2} + 3x = 3x$$

Hence

$$y_p = -\frac{11}{12} - \frac{1}{2}x$$

is a particular solution of the non-homogeneous equation

$$\frac{d^3 y}{dx^3} - 6\frac{d^2 y}{dx^2} + 11\frac{dy}{dx} - 6y = 3x$$

Now consider

$$y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

Then

$$\frac{dy_c}{dx} = c_1 e^x + 2c_2 e^{2x} + 3c_3 e^{3x}$$
$$\frac{d^2 y_c}{dx^2} = c_1 e^x + 4c_2 e^{2x} + 9c_3 e^{3x}$$
$$\frac{d^3 y_c}{dx^3} = c_1 e^x + 8c_2 e^{2x} + 27c_3 e^{3x}$$

Since,

$$\frac{d^3 y_c}{dx^3} - 6\frac{d^2 y_c}{dx^2} + 11\frac{dy_c}{dx} - 6y_c$$
$$= c_1 e^x + 8c_2 e^{2x} + 27c_3 e^{3x} - 6\left(c_1 e^x + 4c_2 e^{2x} + 9c_3 e^{3x}\right)$$
$$+ 11\left(c_1 e^x + 2c_2 e^{2x} + 3c_3 e^{3x}\right) - 6\left(c_1 e^x + c_2 e^{2x} + c_3 e^{3x}\right)$$
$$= 12c_1 e^x - 12c_1 e^x + 30c_2 e^{2x} - 30c_2 e^{2x} + 60c_3 e^{3x} - 60c_3 e^{3x}$$
$$= 0$$

Thus $y_c$ is general solution of associated homogeneous differential equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$$

Hence general solution of the non-homogeneous equation is

$$y = y_c + y_p = c_1e^x + c_2e^{2x} + c_3e^{3x} - \frac{11}{12} - \frac{1}{2}x$$

**Superposition Principle for Non-homogeneous Equations**

Suppose that

$$y_{p_1}, y_{p_2}, \ldots, y_{p_k}$$

denote the particular solutions of the $k$ differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_i(x),$$

$i = 1,2,\ldots k$, on an interval $I$. Then

$$y_p = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x)$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x)$$

**Example**

Consider the differential equation

$$y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^x - e^x$$

Suppose that

$$y_{p_1} = -4x^2, \qquad y_{p_2} = e^{2x}, \qquad y_{p_3} = xe^x$$

Then $\quad y''_{p_1} - 3y'_{p_1} + 4y_{p_1} = -8 + 24x - 16x^2$

Therefore $\quad y_{p_1} = -4x^2$

is a particular solution of the non-homogenous differential equation

$$y'' - 3y' + 4y = -16x^2 + 24x - 8$$

Similarly, it can be verified that

$$y_{p_2} = e^{2x} \text{ and } y_{p_3} = xe^x$$

are particular solutions of the equations:

$$y'' - 3y' + 4y = 2e^{2x}$$

and $\quad y'' - 3y' + 4y = 2xe^x - e^x$

respectively.

Hence $y_p = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^x$

is a particular solution of the differential equation

$$y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^x - e^x$$

**Exercise**

Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Form the general solution.

1. $y'' - y' - 12y = 0;\ e^{-3x}, e^{4x},\ (-\infty, \infty)$
2. $y'' - 2y' + 5y = 0;\ e^x \cos 2x, e^x \sin 2x,\ (-\infty, \infty)$
3. $x^2 y'' + xy' + y = 0;\ \cos(\ln x), \sin(\ln x),\ (0, \infty)$
4. $4y'' - 4y' + y = 0;\ e^{x/2}, xe^{x/2},\ (-\infty, \infty)$
5. $x^2 y'' - 6xy' + 12y = 0;\ x^3,\ x^4\ (0, \infty)$
6. $y'' - 4y = 0;\ \cosh 2x,\ \sinh 2x,\ (-\infty, \infty)$

Verify that the given two-parameter family of functions is the general solution of the non-homogeneous differential equation on the indicated interval.

7. $y'' + y = \sec x, \quad y = c_1 \cos x + c_2 \sin x + x \sin x + (\cos x)\ln(\cos x),\ (-\pi/2, \pi/2)$.
8. $y'' - 4y' + 4y = 2e^{2x} + 4x - 12, \quad y = c_1 e^{2x} + c_2 xe^{2x} + x^2 e^{2x} + x - 2$
9. $y'' - 7y' + 10y = 24e^x, \quad y = c_1 e^{2x} + c_2 e^{5x} + 6e^x,\ (-\infty, \infty)$
10. $x^2 y'' + 5xy' + y = x^2 - x, \quad y = c_1 x^{-1/2} + c_2 x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x,\ (0, \infty)$