LECTURE 1: ONE-DIMENSIONAL ELASTICITY

1 Introduction: one-dimensional elasticity

■ **Overview** We explore elasticity in one dimension to give a general ideas of the different steps necessary to develop a general theory of elasticity.

1.1 A one-dimensional theory

Here, we consider a one-dimensional continuum that can only deform along its length. Therefore, there is no bending, twisting, or shearing, just stretching. The emphasis here is on understanding the different steps that enter in the development of a full theory of continuum in the simplest possible context. The steps are

1) **Kinematics:** A description of the possible deformations. The definition of *strains*, given by geometry. In oour context, it is just the stretch along the line.

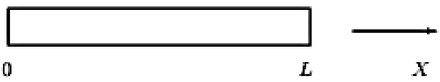
2) **Mechanics:** The definitions of *stresses* and *forces* acting on the medium. Then a statement of balance laws based on the balance of linear and angular momenta, this is applicable to all continuum media but for our problem, linear momentum is sufficient.

3) **Constitutive laws:** A statement of the relationship between stresses and strains. This is where we describe the response of the material under loads.

The results of these three steps is a closed set of equations whose solutions (with appropriate boundary conditions and initial data) is a description of the stresses and deformations in a particular body under a particular set of forces.

1.1.1 Kinematics

Consider a 1D continuum of length L. Any material point is labelled by $X \in [0, L]$. The motion or deformation is the mapping $x = x(X, t)$, which is assumed smooth and invertible as there is no material separation, discontinuity or overlap. The kinematics is fully describe



by a the stretch and the velocity at one point.

$$\lambda = \frac{\partial x}{\partial X}, \quad \text{stretch}; \qquad \dot{x} = V(X,t) = \frac{\partial x}{\partial t}, \quad \text{velocity.}$$

Due to the assumption: $\lambda > 0$, and $x = X$ corresponds to the stress-free (Langrangian) configuration.

Motion: The velocity of a material point is $V(X,t) = \dot{x} = \partial x / \partial t$. Since $X = X(x,t)$ is invertible, we can write,

$$v(x,t) = \dot{x}(X(x,t),t),$$

where v is the velocity at the spatial point x.

The acceleration of a point is,

$$\ddot{x}(X,t) = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}, \qquad \text{or} \qquad a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x},$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x},$$

is the *material time derivative.*

1.2 Dynamics

We use two fundamental principles to obtain equations for the motion of a continuum: the conservation of mass and the balance of linear momentum (in a general theory we will also need the balance of angular momentum).

1.2.1 Conservation of mass

We define ρ to be the linear density in the current configuration (mass per unit length as measured in the current configuration) and ρ_0 the linear density in the reference configuration (as measured in the initial configuration) . Assuming no mass is created, we have

$$\int_{X_1}^{X_2} \rho_0 \,\mathrm{d}x = \int_{x_1}^{x_2} \rho \,\mathrm{d}x,$$

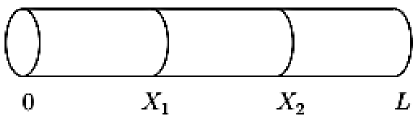
with $x_1 = x(X_1, t), x_2 = x(X_2, t)$. Since $\mathrm{d}x = \lambda \mathrm{d}X$, we have

$$\int_{X_1}^{X_2} \rho_0 \,\mathrm{d}X = \int_{X_1}^{X_2} \rho\lambda \,\mathrm{d}X,$$

which implies that $\lambda\rho = \rho_0$, the Lagrangian conservation of mass. This is the first conservation law.

1.2.2 Balance of linear momentum

The general principle for the Balance of linear momentum is $\frac{\mathrm{d}}{\mathrm{d}t}$(linear momentum) = force acting on the system. Let's break this into the following pieces:



1) The linear momentum:

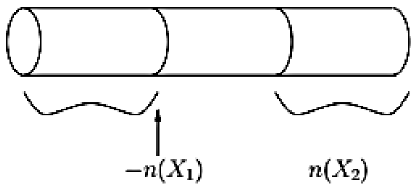
$$\int_{X_1}^{X_2} \rho_0 \dot{x} \,\mathrm{d}X$$

2) forces: themselves due to external (body) forces or internal (contact) forces:

   - body forces,

$$\int_{X_1}^{X_2} \rho_0 f \,\mathrm{d}X$$

   where$f$ is the density of body force (force per unit mass).



- contact forces: force the material exerts on itself.

  This material exerts a force $n(X_2)$ on $[0, X_2]$ counted positive (tensile) if the force is in the direction of the axis, compressive otherwise. Therefore, from action=reaction, the contact force acting on the segment $[X_1, X_2]$ is $n(X_2) - n(X_1)$.

Therefore, the Balance of linear momentum for a one-dimensional continuum implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{X_1}^{X_2} \rho_0 \dot{x}\,\mathrm{d}X = \int_{X_1}^{X_2} \rho_0 f\,\mathrm{d}X + n(X_2) - n(X_1)$$

We can obtain an expression with a single integral by moving the derivative inside the integral and using,

$$\int_{X_1}^{X_2} \frac{\partial n}{\partial X}\,\mathrm{d}X = n(X_2) - n(X_1).$$

That is

$$\int_{X_1}^{X_2} \left(\rho_0 \ddot{x}\,\mathrm{d}X - \rho_0 f \frac{\partial n}{\partial X}\right) \mathrm{d}X = 0.$$

This relation is valid $\forall\ X_1,\ X_2$, so that, we can localise the integral (assuming continuity of the integrand) to obtain

$$\rho_0 a = \rho_0 f + \frac{\partial n}{\partial X}.$$

This is an equation for the force $n(X)$ in the material (Cauchy first equation). This equation is in the reference configuration (all quantities depend on the material variable X and time t). We can obtain an equation in the current configuration by using $\mathrm{d}X = \lambda^{-1}\,\mathrm{d}x$

$$\rho a = \rho f + \frac{\partial n}{\partial x}.$$

But what is $\partial n/\partial x$? We need a constitutive law to close the system.

1.2.3 Constitutive laws

To close the problem, we need to relate the stresses to the strains, that is a relationship between σ and λ such as Hooke's law

$$\sigma = K(\lambda - 1). \tag{1}$$

This Hookean law is only typically valid for small deformations. For large deformations, we will assume in general that the material is *hyperelastic*, that is the constitutive law derives from a potential Ψ capturing the elastic energy associated with deformation so that

$$n = f(\lambda) = \frac{\partial \Psi}{\partial \lambda}. \tag{2}$$

with the requirement that $f(1) = 0$ and that the derivative of f at $\lambda = 1$ exists. For such systems, the Hooke constant $K = f'(1)$ is then simply the linearised behaviour for small deformations around the stress-free state. The theory of three-dimensional elasticity developed in next Chapter when applied to the uniaxial extension of an incompressible rectangular *neo-Hookean* bar suggests the following nonlinear law

$$n = K/3(\lambda^2 - \lambda^{-1}), \tag{3}$$

Close to $\lambda = 1$, we recover Hooke's law (as shown in Fig 1). More generally, materials

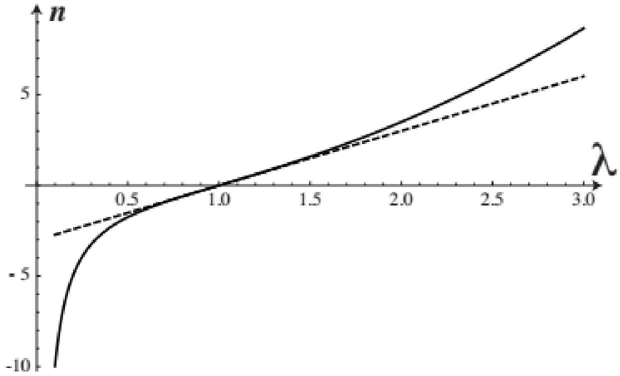


Figure 1: Comparison between the linear (dash) and nonlinear (solid) Hookean response for $K = 3$.

that show strain-stiffening (increase in stiffness for large deformations) or strain-softening (decrease in stiffness) can be modelled by various functions of the stretch.