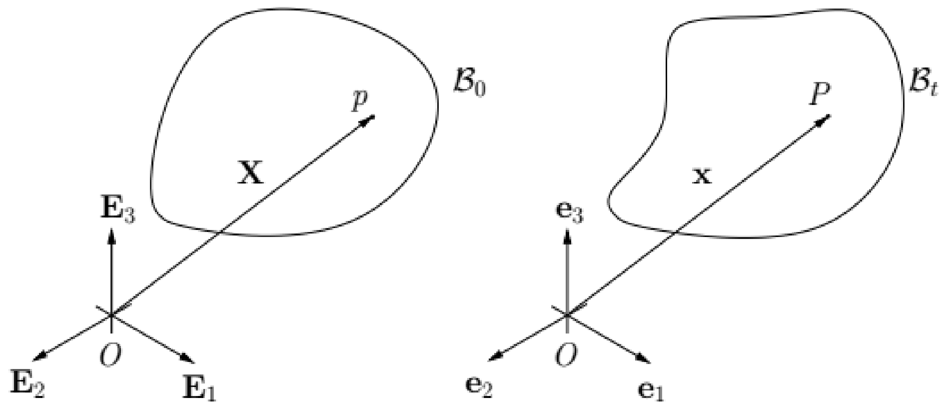
LECTURE 2: KINEMATICS

2 Kinematics

■ **Overview** We develop a completely general theory for the deformation of three-dimensional bodies with no assumptions on displacements. To do so, we introduce two configurations and relate them through the motion in time and the deformation gradient. The deformation gradient is naturally defines as a two-point tensors and its analysis require some tensor calculus.

2.1 Bodies, configuration, deformation

A *body*, B: set whose elements can be put into 1-1 correspondence with points in a region $\mathcal{B} \subset \mathbb{E}^3$. we define *material points* as the elements of B. Since the body moves or deforms it can change with time, $t \in \mathbb{R}$. We denote by $\mathcal{B}_t$ the *configuration* of B at time t. (In particular if we look at static systems, we will use $\mathcal{B}_0$ for the initial configuration and $\mathcal{B}$ for the current one.) Possible terminology for $\mathcal{B}_0$ include initial/reference/material/Lagrangian configuration (here we will use the terminology *reference configuration*). Similarly for $\mathcal{B}_t$, you will find the name current/actual/Eulerian/instantaneous configuration (we will use the terminology *current configuration*).

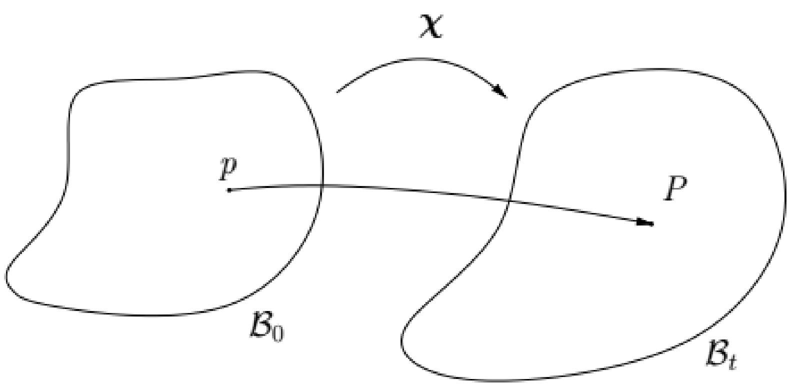


Since both $\mathcal{B}_0$, $\mathcal{B}_t$ are bijections of $B \implies \exists \mathbf{x}$ such that

$$\mathbf{x} = \chi(\mathbf{X}, t), \quad \forall \mathbf{X} \in \mathcal{B}_0 \qquad \text{and} \qquad \mathbf{X} = \chi^{-1}(\mathbf{x}, t), \quad \forall \mathbf{x} \in \mathcal{B}_t,$$

where χ is the *deformation* of $\mathcal{B}$ from $\mathcal{B}_0$ to $\mathcal{B}_t$ (also sometimes called motion.) This construction can be summarised as follows:

Continuum assumption. We consider a body with reference configuration $\mathcal{B}_0 \subset \mathbb{R}^3$. At



time t, the body occupies the current configuration $\mathcal{B}_t \subset \mathbb{R}^3$. A material point, initially at $\mathbf{X} \in \mathcal{B}_0$ is mapped to a point $\mathbf{x} \in \mathcal{B}_t$ by the one-parameter mapping $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ so that $\boldsymbol{\chi} : \mathcal{B}_0 \to \mathcal{B}_t$. The continuum assumption states that $\boldsymbol{\chi}$ is a orientation-preserving bijection mapping for all time t (except possibly at the boundary for contact-problem). This implies that we can write $\mathbf{x} = \boldsymbol{\chi}^{-1}(\mathbf{X})$. We further assume that this mapping is twice continuously differentiable in $\mathbf{X}$ and t. This last assumption can be relaxed in problems involving phase boundaries (with possible jumps in the first derivative). In many instances and applications, we will assume that $\boldsymbol{\chi}$ is actually smooth.

Example Rigid motion.

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) = \mathbf{c}(t) + \mathbf{Q}(t).\mathbf{X}$$

Here $\mathbf{c}$ is a vector and $\mathbf{Q}$ is a proper orthogonal second-order tensor. Using Cartesian coordinates $\mathbf{Q}$ is a rotation matrix (a proper orthogonal matrix, that is a member of the special orthogonal group $SO(3)$.)

$$\mathbf{x} = x_i \mathbf{e}_i, \qquad \mathbf{X} = X_i \mathbf{E}_i,$$

where we use the summation convention and i takes the values 1,2,3. Now choose $\mathbf{E}_i = \mathbf{e}_i$, then

$$x_i = c_i(t) + Q_{ij}(t) X_j,$$

where $\mathbf{c}$ represents a translation and $\mathbf{Q}$ represents a rotation.

Next, we wish to attach physical quantities to every point: mass (a scalar); velocity, traction (vectors); strain/stress (tensors),. . .. These different quantities are defined on all points of the body and are therefore called *fields*. Elasticity is a theory of fields (similar to electromagnetism, relativity, and fluid mechanics).

2.2 Velocities

Velocity of a point P:

$$\mathbf{V} = \dot{\mathbf{x}} \equiv \frac{\partial}{\partial t}\boldsymbol{\chi}(\mathbf{X}, t), \qquad \text{velocity}$$
$$\mathbf{A} = \dot{\mathbf{V}} \equiv \frac{\partial^2}{\partial t^2}\boldsymbol{\chi}(\mathbf{X}, t) \qquad \text{acceleration}$$

Now define a field (for instance, temperature or mass) at every point on B at time t,

$$\phi(\mathbf{x}, t) = \phi(\boldsymbol{\chi}(\mathbf{X}, t), t) = \Phi(\mathbf{X}, t).$$

This change of variable allows us to define this scalar field on the reference configuration.

We are interested in the rate of change of ϕ at a given material point P (fixed $\mathbf{X}$),

$$\dot{\phi} = \frac{\partial \Phi}{\partial t}(\mathbf{X}, t),$$

but

$$\begin{aligned}\frac{\partial \Phi}{\partial t}(\mathbf{X}, t) &= \frac{\partial}{\partial t}\phi(\mathbf{x}, t) + \frac{\partial x_i}{\partial t}\frac{\partial}{\partial x_i}\phi(\mathbf{x}, t)\\ &= \frac{\partial}{\partial t}\phi(\mathbf{x}, t) + \frac{\partial x_i}{\partial t} \cdot \nabla_x \phi(\mathbf{x}, t) = \frac{\mathrm{D}\phi}{\mathrm{D}t},\end{aligned}$$

where we have introduced $\mathrm{D}/\mathrm{D}t$, the *material time derivative*,

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} = \frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla_x \phi,$$

and where $(\nabla_x)_i \equiv \frac{\partial}{\partial x_i}$. Similarly, by applying the same idea component by component, we have for a vector field $\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(\mathbf{X}(\mathbf{x}, \mathbf{t}), \mathbf{t})$

$$\frac{\partial}{\partial t}\mathbf{U}(\mathbf{X}, t) \equiv \dot{\mathbf{U}} \equiv \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{v} \cdot \nabla_x)\mathbf{u}.$$

In particular

$$\mathbf{A} = \dot{\mathbf{V}} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla_x)\mathbf{v}.$$

Note that there is no possible confusion if all the derivatives are taken in the reference configuration since $\mathbf{X}$, t are truly independent. Expressing $\mathbf{X}$ as a function of $(\mathbf{x}, t)$ introduces the extra convective derivative.

To make any progress in the description of deformation we need to define the relative change of length in all possible direction of space. That is we need to define a quantity of the form

$$\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{F},$$

which naturally defines $\mathbf{F}$ as a the gradient of a vector, that is a tensor. Therefore, before proceeding with a proper definition of the deformation gradient, we need to review some basic notions of tensors and tensor calculus.

2.3 A digression about tensors

2.3.1 Just vectors and tensors

We consider a Euclidean vector space in 3D. We will first restrict our attention to the usual Cartesion orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, for which we have $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. We define first vectors

$$\mathbf{u} = u_i \mathbf{e}_i \quad \Longleftrightarrow \quad \mathbf{u} \cdot \mathbf{e}_i = u_i$$

and the space V as the vector space associated with all such vectors in $\mathbf{E}^3$. A *tensor* is then simply a linear map from $V \to V$. That is $\mathbf{S}$ is a tensor $\implies$ $\mathbf{v} = \mathbf{S}\mathbf{u}$, another vector.

One can think of tensors as matrices. But one must be careful as tensors can conveniently be written in a different non-Cartesian basis. So we really want to define operation on tensors independent of the basis and carry the basis as part of the definition of the tensor. To properly define tensors, we need the tensor product.

2.3.2 Tensor product

$\mathbf{u} \otimes \mathbf{v}$ of two vectors $\mathbf{u}$ and $\mathbf{v}$ is a tensor such that

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{a} = (\mathbf{v} \cdot \mathbf{a})\mathbf{u}.$$

In components, $\mathbf{u} = u_i \mathbf{e}_i$ and $\mathbf{v} = v_i \mathbf{e}_i$ so that,

$$\begin{aligned} \mathbf{u} \otimes \mathbf{v} &= u_i \mathbf{e}_i \otimes v_j \mathbf{e}_j \\ &= u_i v_j \mathbf{e}_i \otimes \mathbf{e}_j. \end{aligned}$$

Denoting the components of a tensor

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \quad \iff \quad T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j,$$

we have

$$(\mathbf{u} \otimes \mathbf{v})_{ij} = u_i v_j.$$

After one has defined the matrix $[T_{ij}]$, all known linear algebra identities carry over to tensors.

$$\det \mathbf{T} = \det([T_{ij}])$$
$$\text{tr } \mathbf{T} = \text{tr } ([T_{ij}])$$
$$\mathbf{T}^t = \mathbf{T} \iff T_{ij} = T_{ji}$$

Similarly $\mathbf{ST}$ is defined as $(\mathbf{ST})\mathbf{v} = \mathbf{S}(\mathbf{Tv})$ and not surprising,

$$[ST] = [S][T]$$

Note: I use $\cdot$ for inner product and not for contraction, contraction is assumed in the product of tensors unless otherwise specified.

2.3.3 Derivatives of tensors

We need to define basic operations on tensors. Let ϕ, u, $\mathbf{T}$ be scalar, vector and tensor fields respectively over x,

$$\phi = \phi(\mathbf{x}), \qquad \mathbf{u} = u_i(\mathbf{x})\mathbf{e}_i, \qquad \mathbf{T} = T_{ij}(\mathbf{x})\mathbf{e}_i \otimes \mathbf{e}_j.$$

We define ∇_x or $\partial/\partial\mathbf{x}$ by,

$$\text{grad } \phi = \frac{\partial \phi}{\partial \mathbf{x}} = \frac{\partial \phi}{\partial x_i}\mathbf{e}_i,$$
$$\text{grad } \mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\partial \mathbf{u}}{\partial x_j} \otimes \mathbf{e}_j = \nabla \otimes \mathbf{u}$$
$$= \frac{\partial (u_i \mathbf{e}_i)}{\partial x_j} \otimes \mathbf{e}_j = \frac{\partial u_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j.$$

We recognise $\partial u_i/\partial x_j$ as the Jacobian matrix here.

$$\text{grad } \mathbf{T} = \nabla \otimes \mathbf{T}$$
$$= \frac{\partial}{\partial x_k}(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) \otimes \mathbf{e}_k$$
$$= \frac{\partial T_{ij}}{\partial x_k}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k.$$

We can also contract tensors,

$$\text{div } \mathbf{T} = \frac{\partial T_{ij}}{\partial x_k}\mathbf{e}_j\,(\mathbf{e}_i \cdot \mathbf{e}_j)\,.$$

This is the contraction on the first index of T_i. with index k (Note: this is not a universal choice and some authors contract w.r.t. the second index). Since $\mathbf{e}_i \cdot \mathbf{e}_k = \delta_{ik}$,

$$\text{div } \mathbf{T} = \frac{\partial T_{ij}}{\partial x_i}\mathbf{e}_j, \qquad \text{a vector.}$$

So far, we have defined tensor in a single basis. However, the deformation gradient takes the derivative of a vector in one basis w.r.t. a vector in another basis, in which case we have mixed basis.