

LECTURE 4: CONSERVATION LAWS, STRESS AND DYNAMICS

■ **Overview** We use basic physical principles to derive local equations for the evolution of mass and stress in space. The requirement for localising the balance of linear momentum naturally leads to the definition of the Cauchy stress tensor.

3.1 Mass conservation

We start with a scalar field ρ , the *mass density* defined at each point of the body. Assuming that mass is additive and $m \rightarrow 0$ as the volume of $\Omega \rightarrow 0$, the mass of a given reference volume Ω is

$$m(\Omega) = \int_{\Omega} \rho \, dv.$$

We *postulate* conservation of mass,

$$\frac{d}{dt} m(\Omega) = 0 = \frac{d}{dt} \int_{\Omega} \rho \, dv.$$

Since Ω changes during the deformation, the derivative cannot be taken directly inside the integral. Instead, we map the integral back to the reference configuration. That is, we apply a transport formula,

$$\frac{d}{dt} \int_{\Omega_0} \rho J \, dV = 0 \implies \int_{\Omega_0} \frac{d}{dt} (\rho J) \, dV = 0 \implies \int (J \dot{\rho} + J \rho \operatorname{div} \mathbf{v}) \, dV = 0.$$

This implies the continuity equation,

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0.$$

This is an example of transport and localization. An alternative way to derive this result is to define the reference density ρ_0 and realise that

$$\rho_0 = \rho J \implies \dot{\rho}_0 = 0 = J (\dot{\rho} + \rho \operatorname{div} \mathbf{v}).$$

where we have used $\dot{\mathbf{J}} = \mathbf{J} \operatorname{div} \mathbf{v}$.

3.2 Body and surface forces

Before proceeding with the Balance of linear and angular momenta we discuss the forces and torque acting in a body.. Consider $\Omega \in \mathcal{B}$ and define $\mathbf{F}(\Omega)$ a force and $\mathbf{G}(\Omega, \mathbf{0})$ a torque with respect to a fixed point $\mathbf{0}$. Both force and torque act on Ω . \mathbf{F} is due to body forces (external) and contact forces (internal).

3.2.1 Body forces

$$\begin{aligned} \mathbf{F}_{\text{body}} &= \int_{\Omega} \rho \mathbf{b} \, dv, & \mathbf{b} & \text{body force per unit mass} \\ \mathbf{G}_{\text{body}} &= \int_{\Omega} (\mathbf{x} \times (\rho \mathbf{b}) + \rho \mathbf{c}) \, dv, & \mathbf{c} & \text{body torque per unit mass} \end{aligned}$$

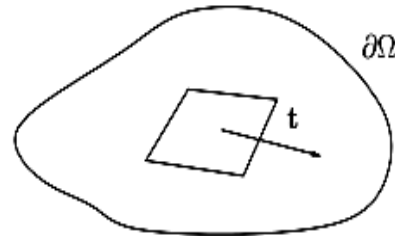
3.2.2 Contact forces

$$\mathbf{F}_{\text{contact}}(\Omega) = \int_{\partial\Omega} \mathbf{t} \, da$$

$$\mathbf{G}_{\text{contact}}(\Omega) = \int_{\partial\Omega} \mathbf{x} \times \mathbf{t} \, da$$

The action of the body outside $\partial\Omega$ on Ω is given by \mathbf{t} , and we use Cauchy's stress principle that \mathbf{t} depends smoothly on \mathbf{n} (for a non-polar medium).

We assume that there is no contact torque.
 Note \mathbf{t} : traction vector force per unit area.



3.3 Momenta and Euler's laws

The linear momentum is defined as

$$\mathbf{M}(\Omega) = \int_{\Omega} \rho \mathbf{v} \, dv.$$

The angular momentum with respect to $\mathbf{0}$ is

$$\mathbf{H}(\Omega; \mathbf{0}) = \int_{\Omega} \mathbf{x} \times (\rho \mathbf{v}) \, dv.$$

3.4 Euler's law of motion

The laws of motion for a body simply state that the rate of change of linear momentum of an arbitrary material subset $\Omega_t \subset \mathcal{B}_t$ is equal to sum of all the forces acting on Ω_t . Similarly, the rate of change of angular momentum of an arbitrary material subset $\Omega_t \subset \mathcal{B}_t$ with respect to a given point is equal to sum of all the torques acting on Ω_t with respect to the same point.

$$\frac{d\mathbf{M}}{dt} = \mathbf{F}, \quad \frac{d\mathbf{H}}{dt} = \mathbf{G}$$

Interestingly in classical discrete point mechanics $d\mathbf{H}/dt = \mathbf{G}$ is a consequence of $d\mathbf{M}/dt = \mathbf{F}$. However, for continuum bodies, these appear as distinct laws.

Postulate: we further assume that we have non-polar media, that is the body is not subject to body or contact torques which in turn implies that $\mathbf{c} = 0$. This leaves us with the following two equations,

$$\frac{d}{dt} \int_{\Omega} \rho \mathbf{v} \, dv = \int_{\Omega} \rho \mathbf{b} \, dv + \int_{\partial\Omega} \mathbf{t} \, da, \quad (*)$$

$$\frac{d}{dt} \int_{\Omega} \rho \mathbf{x} \times \mathbf{v} \, dv = \int_{\Omega} \rho \mathbf{x} \times \mathbf{b} \, dv + \int_{\partial\Omega} \mathbf{x} \times \mathbf{t} \, da.$$

Again we apply the idea of transport:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \rho \mathbf{v} dv &= \frac{d}{dt} \int_{\Omega_0} \rho \mathbf{v} J dV \\ &= \int_{\Omega_0} \frac{d}{dt} (\rho \mathbf{v} J) dV \\ &= \int_{\Omega_0} (J \dot{\rho} \mathbf{v} + J \dot{\rho} \mathbf{v} + \rho \mathbf{v} J \operatorname{div} \mathbf{v}) dV \\ &= \int_{\Omega} \rho \dot{\mathbf{v}} + \underbrace{\dot{\rho} \mathbf{v} + \rho \mathbf{v} \operatorname{div} \mathbf{v}}_{=0, \text{ continuity}} dV \\ &= \int_{\Omega} \rho \dot{\mathbf{v}} dV \end{aligned}$$

Therefore (*) reads

$$\begin{aligned} \int_{\Omega} \rho(\mathbf{a} - \mathbf{b}) dv &= \int_{\partial\Omega} \mathbf{t} da, \\ \int_{\Omega} \rho \mathbf{x} \times (\mathbf{a} - \mathbf{b}) dv &= \int_{\partial\Omega} \mathbf{x} \times \mathbf{t} da. \end{aligned}$$

Now a fundamental problem: if we want to localize this integral, we need to express

$$\int_{\partial\Omega} \mathbf{t} da \quad \text{as} \quad \int_{\Omega} (?) dv$$

We could use the divergence theorem:

$$\int_{\Omega} \operatorname{div} \mathbf{v} = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} da,$$

for any closed $\partial\Omega \in \mathbb{R}^3$, but then we need $\mathbf{v} \cdot \mathbf{n} da$ not $\mathbf{t} da$.

This consideration naturally leads us to introduce a new fundamental object for the description of forces in a body, the stress. Cauchy's theorem states that if

$$\int_{\Omega} \rho(\mathbf{a} - \mathbf{b}) dv = \int_{\partial\Omega} \mathbf{t} da,$$

then \exists a second-order tensor, the Cauchy stress tensor, such that

$$\mathbf{t} = \mathbf{T}^t \mathbf{n}, \quad \text{Cauchy's theorem.}$$

We are now able to express all quantities appearing in the balance of linear momentum as a single integral

$$\int_{\Omega} \rho(\mathbf{a} - \mathbf{b}) dv = \int_{\partial\Omega} \mathbf{t} da = \int_{\partial\Omega} \mathbf{T}^t \mathbf{n} da = \int_{\Omega} \operatorname{div} \mathbf{T} dv,$$

and we can localise this integral to obtain

$$\int_{\Omega} (\rho(\mathbf{a} - \mathbf{b}) - \operatorname{div} \mathbf{T}) dv = 0,$$

$$\Rightarrow \boxed{\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \rho \mathbf{a}}$$

The balance of angular momentum follows the same basic steps and leads to a restriction on the Cauchy stress tensor

$$\boxed{\mathbf{T}^t = \mathbf{T}}$$

That is, \mathbf{T} is a symmetric tensor.