LECTURE 5: DIGRESSION: THE DIVERGENCE THEOREM FOR TENSORS

Let $\mathbf{v}$ be a vector field. Then

$$\int_\Omega \operatorname{div} \mathbf{v}\, \mathrm{d}V = \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\, \mathrm{d}A \qquad \Longleftrightarrow \qquad \int_\Omega \nabla\phi\, \mathrm{d}V = \int_{\partial\Omega} \phi\mathbf{n}\, \mathrm{d}A.$$

(To see this, take $\mathbf{v} = \mathbf{c}\phi$ for $\mathbf{c} \neq 0$, constant.) This implies

$$\int_\Omega \frac{\partial\phi}{\partial x_i} \mathrm{d}V = \int_{\partial\Omega} \phi n_i\, \mathrm{d}A.$$

Apply this to each component of a Cartesian tensor, $T_{jk\ell\ldots}$, then

$$\int_\Omega \frac{\partial T_{jk\ell\ldots}}{\partial x_i} \mathrm{d}V = \int_{\partial\Omega} T_{jk\ell\ldots} n_i\, \mathrm{d}A,$$

In particular

$$\int_\Omega \operatorname{div} \mathbf{T}\, \mathrm{d}V = \int_{\partial\Omega} \mathbf{T}^T\mathbf{n}\, \mathrm{d}A,$$

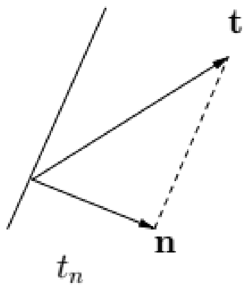
or in index notation,

$$\int_\Omega \frac{\partial T_{ij}}{\partial x_i} \mathrm{d}V = \int_{\partial\Omega} T_{ij} n_i\, \mathrm{d}A,$$

3.4.2 More on the stress tensor

Consider again the stress tensor $\mathbf{T}$

- normal stress: $t_n = \mathbf{n}\cdot(\mathbf{T}\mathbf{n}) = \mathbf{n}\cdot\mathbf{t}$
  If $t_n > 0$ then this stress is tensile, and if $t_n < 0$ it is compressive.
- shear stress: $t_s = |\mathbf{t} - \mathbf{T}\mathbf{n}|$

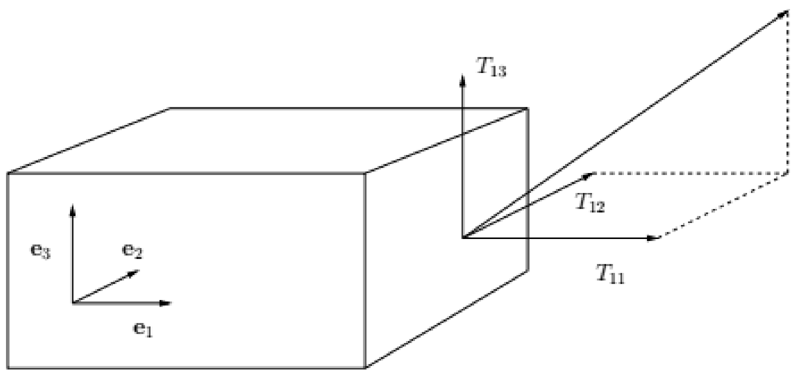


Particular case Suppose $t_s = 0$ and t_n is independent of $\mathbf{n}$. Then

$$\mathbf{t} = -p\mathbf{n}, \qquad \sigma = -p\mathbb{1}.$$

This is hydrostatic stress.

T_{ij} is the j^{th} component of the force in the i^{th} direction.



T_{11}: normal stress; T_{12}, T_{13}: shear stresses.

3.4.3 The nominal and Piola Kirchhoff stress tensors

The Cauchy stress tensor is defined in the current configuration. We can ask the following question, which tensor defined in the reference configuration gives the traction when applied to an local area element? To answer this question we usese Nanson's formula:

$$\mathbf{n}\,\mathrm{d}a = \mathbf{J}\mathbf{F}^{-t}\mathbf{N}\,\mathrm{d}A$$

and apply it to the traction vector

$$\Longrightarrow \quad \mathbf{t}\,\mathrm{d}a = \mathbf{T}\mathbf{n}\,\mathrm{d}a = (\mathbf{J}\mathbf{T}\mathbf{F}^{-t})\mathbf{N}\,\mathrm{d}A,$$

where $\mathbf{S}$ is the nominal stress tensor,

$$\boxed{\mathbf{S} = \mathbf{J}\mathbf{F}^{-1}\mathbf{T}}$$

$\mathbf{S}^T$ is the Piola–Kirchhoff force per unit undeformed area.
Since $\mathbf{T}$ is symmetric, $\mathbf{T} = \mathbf{T}^t$,

$$\mathbf{J}^{-1}\mathbf{F}^T\mathbf{S}^T = \mathbf{J}^{-1}\mathbf{F}\mathbf{S},$$

$$\Longrightarrow \quad \boxed{\mathbf{F}^T\mathbf{S}^T = \mathbf{F}\mathbf{S}}$$