LECTURE 6: CONSTITUTIVE EQUATIONS

■ **Overview** We close the system of Cauchy equations for stress, density, and velocity by introducing a relationship between stresses and strains, the constitutive equations. Depending on the choice of constitutive equations, the continuum equations can describe a fluid, a solid, or a gas.

So far, we have obtained through conservation of mass and balance of momenta, the following three equations

$$\begin{aligned} \dot{\rho} + \rho \text{div } \mathbf{v} &= 0, && \text{mass} \\ \text{div } \mathbf{T} + \rho \mathbf{b} &= \rho \dot{\mathbf{v}}, && \text{linear momentum} \\ \mathbf{T}^t &= \mathbf{T}, && \text{angular momentum} \end{aligned}$$

There are 10 unknowns: 1 in ρ, 3 in vector $\mathbf{v}$ and 6 in the symmetric tensor $\mathbf{T}$. But there are only 4 equations. We need 6 extra relationships to close this system. These will be given by the constitutive equations.

4.1 3 types of assumptions

1) *Possible deformations.*
   e.g. Only rigid motions are allowed ($\mathbf{F} = \mathbf{R}$, 3 parameters). $\Longrightarrow$ rigid body mechanics.
   e.g. Only isochoric motion $\Longrightarrow$ Incompressible material.

2) *Constraining the stress tensor*
   e.g. $\mathbf{T} = \mathcal{T}(\mathbf{F})$
   e.g. $\mathbf{T} = -p\mathbb{1}$

3) *Relate stress to motion*
   e.g. pressure function of density, ρ (for a gas).

4.1.1 Particular examples

1) *Ideal fluids*

   (a) $\det \mathbf{F} = 1$ (Isochoric)

   (b) $\rho = \text{const}$

   (c) $\mathbf{T} = -p\mathbb{1}$

   Note: the pressure is *not* determined by the motion (ball under uniform pressure). (Lagrange multiplier for the pressure.)

2) *Elastic fluids*

   (a) $\mathbf{T} = -p\mathbb{1}$

   (b) $p = p(\rho)$

   Here $\ddot{\rho} = P'(\rho_0)\Delta\rho$ and $\sqrt{p'}$ is the sound speed.
   N.B.: both fluids are inviscid (do not exert shearing forces!)

   A particular case of an elastic fluids is an *ideal gas*: $p = \lambda\rho^\gamma$, for $\lambda > 0$, $\gamma > 1$.

3) *Newtonian fluids.* Shear stress through friction.
   Take $\mathbf{L} = \text{grad } \mathbf{v}$ which gives relative motion of particles, velocity gradient.

(a) $\det \mathbf{F} = 1$, incompressible
(b) $\mathbf{T} = -p\mathbb{1} + \mathcal{C}[\mathbf{L}]$ where $\mathcal{C}$ is a linear function of $\mathbf{L}$.

Note $\mathcal{C}[0] = 0 \implies \mathbf{T} = -p\mathbb{1}$, A Newtonian fluid at rest is ideal
Note $\mathcal{C}[\mathbf{L}]$ has 40 independent constants (once we have removed arbitrariness of $p\mathbb{1}$. However *objectivity* (independence from observer) implies

$$\mathcal{C}[\mathbf{L}] = 2\mu\mathbf{D}, \qquad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T),$$

which has a single constant, viscosity μ. This implies

$$\begin{aligned} \rho\dot{\mathbf{v}} &= \text{div } \mathbf{T} + \rho\mathbf{b} \\ \text{div } \mathbf{v} &= 0 \\ \mathbf{T} &= -p\mathbb{1} + 2\mu\mathbf{D} \end{aligned}$$

After some algebra,

$$\begin{aligned} \rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v} \cdot \text{grad } \mathbf{v} &= \mu\Delta\mathbf{v} - \text{grad } p + \rho\mathbf{b}, \\ \text{div } \mathbf{v} &= 0, \end{aligned}$$

which are the Navier–Stokes equations. (N.B. $\nu = \mu/\rho$ is the kinematic viscosity.)
Stokes flow: 1) steady, 2) neglect acceleration.

$$\begin{aligned} \Delta\mathbf{v} &= \text{grad } p - \mathbf{b} \\ \text{div } \mathbf{v} &= 0. \end{aligned}$$

N.B. for more general fluids, $\mathbf{T} = -p\mathbb{1} + \mathcal{N}(\mathbf{L})$.

4.2 Elastic materials

For elastic materials, we have the simple relationship

$$\mathbf{T} = \mathcal{Z}(\mathbf{F})$$

This implies that the stress in $\mathcal{B}$ at $\mathbf{x}$ depends on $\mathbf{F}$ and not on the history of the deformation (path-independent). Also, by the definition of the reference configuration (assuming that it is stress free), we have

$$\mathcal{Z}(\mathbb{1}) = 0.$$

This relationship defines *a Cauchy, elastic material.*

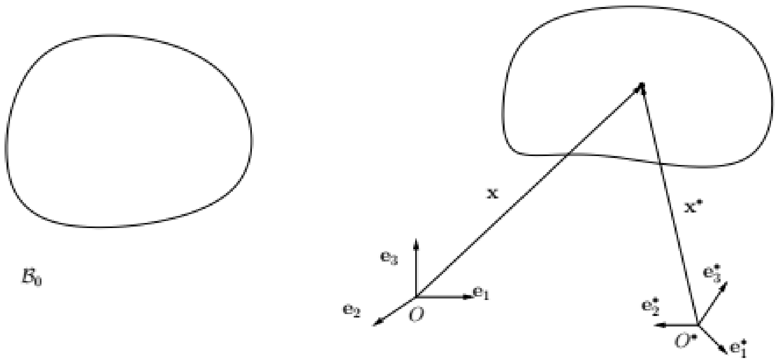
4.2.1 Objectivity or frame indifference

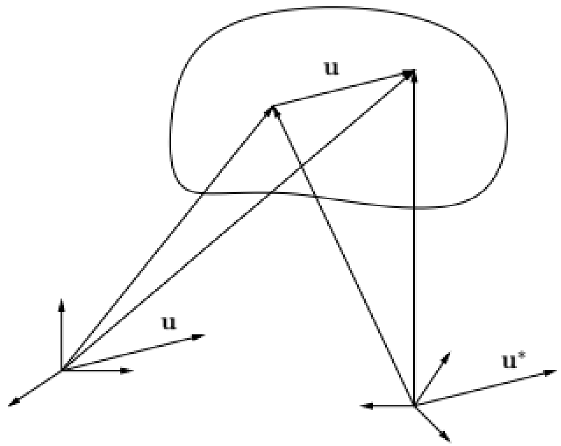
Material properties are independent of the frame in which they are observed (or the observer).

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \qquad \mathbf{x}^* = \boldsymbol{\chi}^*(\mathbf{X}, t), \qquad t^* = t.$$

The two descriptions are related by

$$\mathbf{x}^* = \mathbf{Q} \cdot \mathbf{x} + \mathbf{c},$$





where $\mathbf{Q} = \mathbf{Q}(t)$ is orthonormal and $\mathbf{c} = \mathbf{c}(t)$.

$$\mathbf{F}^* = \frac{\partial \mathbf{x}^*}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}^*}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{QF}.$$

$$\mathbf{u} = \mathbf{y} - \mathbf{x}, \qquad \mathbf{u}^* = \mathbf{y}^* - \mathbf{x}^* = \mathbf{Q}(\mathbf{y} - \mathbf{x}) = \mathbf{Qu}$$

Therefore

$$\boxed{\mathbf{u} \text{ is objective if } \mathbf{u}^* = \mathbf{Qu}}.$$

If $\mathbf{u}$ is the traction vector or normal

$$\mathbf{t}^* = \mathbf{Qt}, \qquad \mathbf{n}^*\mathbf{Qn}$$

but

$$\mathbf{t} = \mathbf{Tn}, \qquad \mathbf{t}^* = \mathbf{T}^*\mathbf{n}^*,$$

which implies

$$\mathbf{QTn}^* = \mathbf{T}^*\mathbf{n}^* = \mathbf{T}^*\mathbf{Qn}.$$

This is true $\forall \mathbf{n}$, which implies $\mathbf{T}^*\mathbf{Q} = \mathbf{QT}$,

$$\boxed{\mathbf{T}^* = \mathbf{QTQ}^T}$$

More generally a tensor $\mathbf{T}$ is objective if

$$\boxed{\mathbf{T}^* = \mathbf{QTQ}^T}$$

which implies that ϕ, $\mathbf{u}$, $\mathbf{T}$ are objective if

$$\phi = \phi^*, \qquad \mathbf{u}^* = \mathbf{Qu}, \qquad \mathbf{T}^* = \mathbf{QTQ}^T.$$

Care! $\mathbf{v}$, $\mathbf{a}$ are not objective (unless $\dot{\mathbf{Q}} = \dot{c} = 0$), but $(\text{grad}\ \phi)^* = \mathbf{Q}\text{grad}\ \phi$ is objective.

4.3 Hyperelastic materials

4.3.1 Energy balance, elastic materials

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \mathcal{P},$$

where $\mathcal{P}$ is the rate of working, the power of the forces.

$$\mathcal{E}(\Omega) = \mathcal{K}(\Omega) + \mathcal{S}(\Omega),$$

where $\mathcal{K}$ is the kinetic energy,

$$\mathcal{K}(\Omega) = \int_\Omega \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\,\mathrm{d}v,$$

and $\mathcal{S}$ is the internal energy of the system

$$\mathcal{S}(\Omega) = \int_\Omega \Psi\,\mathrm{d}v,$$

where Ψ the internal energy density for an isothermal process, *i.e.*assuming no dissipation.

$$\mathcal{P}(\Omega) = \int_\Omega \rho\mathbf{b}\cdot\mathbf{v}\,\mathrm{d}v + \int_{\partial\Omega} \mathbf{t}\cdot\mathbf{v}\,\mathrm{d}a$$

$$\frac{\mathrm{d}\mathcal{K}(\Omega)}{\mathrm{d}t} - \mathcal{P}(\Omega) = \underset{\text{many steps}}{\ldots} = \int_\Omega \mathrm{tr}(\mathbf{TD})\mathrm{d}v, \quad \text{stress power.}$$

(Hint: use $\mathbf{T}^T = \mathbf{T}$, divergence theorem, equation of motion and continuity.)

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Psi\,\mathrm{d}v = \int \mathrm{tr}(\mathbf{TD})\mathrm{d}v$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_0} \underbrace{\mathbf{J}\Psi}_{\mathrm{W}}\,\mathrm{d}V - \int_{\Omega_0} \mathbf{J}\mathrm{tr}(\mathbf{TD})\mathrm{d}v = 0$$

$$\frac{\mathrm{dW}}{\mathrm{d}t} = p \equiv \mathbf{J}\mathrm{tr}(\mathbf{TD}),$$

where p is the stress power.

Assume *hyperelasticity* that $\mathsf{W} = \mathsf{W}(\mathbf{F})$ only.

$$J\mathrm{tr}(\mathbf{TD}) = \mathrm{tr}(\mathbf{S}\dot{\mathbf{F}})$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{W}(\mathbf{F}) = \mathrm{tr}\left(\frac{\partial \mathsf{W}}{\partial \mathbf{F}}\dot{\mathbf{F}}\right),$$

where

$$\frac{\partial \mathsf{W}}{\partial \mathbf{F}} = \frac{\partial \mathsf{W}}{\partial F_{ji}}\mathbf{e}_i \otimes \mathbf{e}_j, \qquad \left(\frac{\partial \mathsf{W}}{\partial \mathbf{F}}\right)_{ij} = \frac{\partial \mathsf{W}}{\partial F_{ji}}.$$

We have

$$\mathrm{tr}\left[\left(\frac{\partial \mathsf{W}}{\partial \mathbf{F}} - \mathbf{S}\right) \cdot \dot{\mathbf{F}}\right] = 0.$$

This must be true for all motion.

$$\Longrightarrow \boxed{\mathbf{S} = \frac{\partial \mathsf{W}}{\partial \mathbf{F}}}$$

where

$$S_{ij} = \frac{\partial \mathsf{W}}{\partial F_{ji}}, \qquad \mathbf{T} = J^{-1}\mathbf{F}\frac{\partial \mathsf{W}}{\partial \mathbf{F}}.$$

An hyperelastic material is one whose stress derives from a scalar energy density function.

4.3.2 Objectivity of W

$$\mathsf{W} \text{ scalar} \implies \mathsf{W}^* = \mathsf{W}.$$
$$\mathbf{F} \text{ gradient} \implies \mathbf{F}^* = \mathbf{QF}.$$

$$\implies \mathsf{W}^*(\mathbf{F}^*) = \mathsf{W}(\mathbf{QF}) = \mathsf{W}(\mathbf{F})$$

$$\implies \mathsf{W}(\mathbf{QF}) = \mathsf{W}(\mathbf{F}), \quad \forall \text{ rotations } \mathbf{Q}.$$

4.3.3 Material symmetry

$$\mathbf{F} = \mathbf{F}'\mathbf{P}$$

Now

$$\mathbf{T} = \mathcal{T}(\mathbf{F}) = \mathcal{T}'(\mathbf{F}') \quad \Longrightarrow \quad \mathcal{T}(\mathbf{F}'\mathbf{P}) = \mathcal{T}'(\mathbf{F}'),$$

in general $\mathcal{T} \neq \mathcal{T}'$.

If $\mathcal{T}(\mathbf{F}'\mathbf{P}) = \mathcal{T}(\mathcal{F}')$, $\forall \mathbf{F}'$, then $\mathbf{P}$ is a symmetry of the body.

The set of all $\mathbf{P}$ for which this is true form a group, the symmetry group of the material.