LECTURE 7: EXAMPLES OF BOUNDARY VALUE PROBLEMS

■ **Overview** For a given strain-energy density function, we can write a full system of equations which can be solved for given boundary conditions. We give some simple solutions for homogeneous and semi-inverse problems.

5.1 Homogeneous deformations

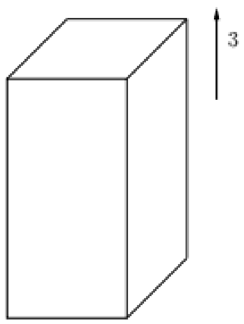
In the compressible case, of the deformation is homogeneous than F = const matrix $\Longrightarrow$ $\frac{\partial \mathsf{W}}{\partial \mathbf{F}}$ constant tensor. Then

$$\mathbf{S} = \frac{\partial \mathsf{W}}{\partial \mathbf{F}} \quad \text{and} \quad \mathbf{T} = J^{-1}\mathbf{F}\frac{\partial \mathsf{W}}{\partial \mathbf{F}}$$

are both constant.

$$\Longrightarrow \text{Div } \mathbf{S} = 0, \qquad \text{div } \mathbf{T} = 0.$$

That is, the Cauchy equations identically satisfied.



Consider, as an example, the diagonal transformation
$\mathbf{T} = \text{diag}(t_{11}, t_{22}, t_{33})$ and $\mathbf{F} = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$

$$\Longrightarrow t_{ii} = \frac{\lambda_i}{J}\frac{\partial \mathsf{W}}{\partial \lambda_i}, \qquad i = 1, 2, 3.$$

The solution is fully specified by the boundary conditions (Ericksen's theorem).

For instance, uniaxial extension is obtained by the choice $t_{33} = N$ and $t_{11} = t_{22} = 0$,

$$\Longrightarrow t_{11} = \frac{1}{\lambda_2\lambda_3}\mathsf{W}_1 = 0, \qquad t_{22} = \frac{1}{\lambda_3\lambda_1}\mathsf{W}_2 = 0, \qquad t_{33} = \frac{1}{\lambda_1\lambda_2}\mathsf{W}_3 = N$$

$$\mathsf{W} = \frac{\mu_1}{2}(I_1 - 3) - \frac{\mu_2}{2}(I_3 - 1) = \frac{\mu_1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) - \frac{\mu_2}{2}(\lambda_1^2\lambda_2^2\lambda_3^2 - 1)$$

$$t_{ii} = \frac{\lambda_i}{J}\frac{\partial \mathsf{W}}{\partial \lambda_i}$$

$$t_{11} = \frac{\lambda_1}{J}\left(\mu_1\lambda_1 - \mu_2\lambda_1\lambda_2^2\lambda_3^3\right) = 0$$
$$t_{22} = \frac{\lambda_2}{J}\left(\mu_1\lambda_2 - \mu_2\lambda_1^2\lambda_2\lambda_3^3\right) = 0$$

$\mu_1 = \mu_2$ (so that $t_{ii} = 0$ when $\lambda_i = 1$).

$$\lambda_1(1-\lambda_2^2\lambda_3^2) = 0$$
$$\lambda_2(1-\lambda_1^2\lambda_3^2) = 0, \qquad J = \lambda^2\lambda^{-1} = \lambda$$
$$\implies \lambda_2^{-2} = \lambda_3^2 = \lambda_1^{-2} \implies \lambda_1 = \lambda_2 = \lambda^{-1}, \quad \lambda_3 = \lambda.$$

$$t_{33} = \mu\left(\frac{\lambda^2}{J} - \lambda^2\frac{\lambda^{-4}}{J}\right) = N$$

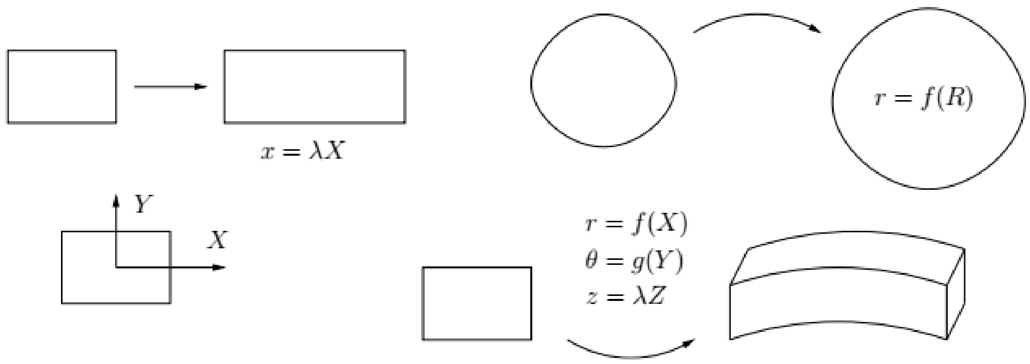
$$J = \frac{\lambda}{\lambda^2} = \lambda^{-1} \qquad \implies \qquad \frac{N}{\mu} = \lambda^3 - \frac{1}{\lambda} = \frac{\lambda^4 - 1}{\lambda}$$

Now

$$\left.\frac{\partial N(\lambda)}{\partial \lambda}\right|_{\lambda=1} = \mu\left.\left(3\lambda + \frac{1}{\lambda^2}\right)\right|_{\lambda=1} = 4\mu.$$

5.2 General method for semi-inverse problems, BVP

1) Describe your material
   - Elastic?
   - Static?
   - Any particular geometry (thin, long...)?
   - Incompressible?
   - Isotropic?
   - Strain - Energy (Define or keep undefined as long as needed.)
2) Describe the deformation.
   - Semi-inverse method.



   - Choose variables in $\mathcal{B}_0$, $\mathcal{B}$ suitable for you problem.
   - Define χ, $\mathbf{F}$, λ_i.
     λ_i is a function of a few parameters, functions, (f, a).

3) Define the boundary conditions. There are loads to choose from! $\mathbf{T}\cdot\mathbf{n}$ on $\partial\mathcal{B}$? Or χ? $\rho\mathbf{b}$?

4) Write down Cauchy equation plus constant.

$$\begin{cases} \text{div } \mathbf{T} + \rho\mathbf{b} = 0 & \text{(static)}, \\ \mathbf{T} = J^{-1}\mathbf{F}\frac{\partial \mathsf{W}}{\partial \mathbf{F}} - p\mathbb{1} \end{cases}$$

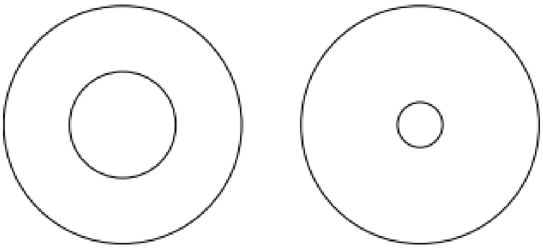
Incompressible is $J = 1$, compressible $p = 0$.
To solve write $\mathsf{W} = \overline{\mathsf{W}}(I_1, I_2, I_3)$, then $\mathsf{W} = \widehat{\mathsf{W}}(\lambda_1, \lambda_2, \lambda_3)$, then $\frac{\partial \mathsf{W}}{\partial \mathbf{F}}$.

$$\implies \mathbf{T} = \mathbf{T}(f(\mathbf{x}), \mathbf{x}), \quad \text{or} \quad \mathbf{S} = \mathbf{S}(f(\mathbf{x}), \mathbf{x}).$$

5) Insert into div $\mathbf{T} + \rho\mathbf{b} = 0$ or Div $\mathbf{S} + \rho_0\mathbf{b} = 0$. Obtain differential equation for $f(\mathbf{x})$ or $p = p(\mathbf{x})$. Solve the equation with the correct boundary conditions.

5.3 Inflation of a spherical shell.

1) Elastic, incompressible, isotropic spherical shell with strain-energy $\mathsf{W}(I_1, I_2, I_3)$.



2) Symmetric inflation

$$A \leqslant R \leqslant R, \qquad \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \quad \text{(see §2.5)}$$
$$\mathbf{x} = f(R)\mathbf{X}, \qquad r = f(R)R.$$

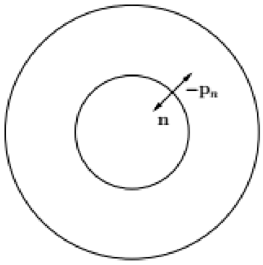
$$\mathbf{F} = \begin{bmatrix} \lambda_r & & \\ & \lambda_\theta & \\ & & \lambda_\phi \end{bmatrix} = \begin{bmatrix} r' & & \\ & r/R & \\ & & r/R \end{bmatrix}.$$

$$\lambda_r = r'(R), \qquad \lambda_\theta = r/R = \lambda_\phi$$
$$\lambda_a = a/A, \qquad \lambda_b = b/B, \qquad r = \sqrt[3]{a^3 - A^3 + R^3}$$

where a is the single unknown parameter. Therefore

$$\lambda_\theta = \lambda_\phi = \lambda = r/R, \qquad \lambda_r = \lambda^{-2}$$

$$\mathbf{T}\cdot\mathbf{n} = \begin{cases} -P & \text{on } r = a \\ 0 & \text{on } r = b \end{cases}$$



$\mathbf{S}^T \cdot \mathbf{N} = JP\mathbf{F}^{-T}\mathbf{N}$ (mapping of traction vector)

$$\begin{cases} \mathbf{Tn} = -P\mathbf{n} & \text{on } \partial\mathcal{B} \\ \mathbf{S}^T \cdot \mathbf{N} = -PJ\mathbf{F}^{-T}\mathbf{N}, & \text{on } \partial\mathcal{B} \end{cases}$$

$$\implies T_{rr} = \begin{cases} -P & \text{on } r = a \\ 0 & \text{on } r = b \end{cases}$$

or

$$S_{rr} = \begin{cases} -P\lambda_r^{-1} = -P\lambda^2 & \text{on } R = A \\ 0 & \text{on } R = B \end{cases}$$

Note that the boundary condition depends on the deformation.

3) $\mathbf{b} = 0$ and div $\mathbf{T} = 0$,

$$\implies \frac{\mathrm{d}T_{rr}}{\mathrm{d}r} + \frac{2}{r}(T_{rr} - T_{\theta\theta}) = 0,$$

or

$$\frac{\mathrm{d}S_{rr}}{\mathrm{d}R} + \frac{2}{R}(S_{rr} - S_{\theta\theta}) = 0.$$

Constitutive equations,

$$\mathbf{S} = \frac{\partial \mathsf{W}}{\partial \mathbf{F}} - p\mathbf{F}^{-1},$$

or

$$\mathbf{T} = \mathbf{F}\frac{\partial \mathsf{W}}{\partial \mathbf{F}} - p\mathbb{1}.$$

Then

$$S_{rr} = \frac{\partial \mathsf{W}}{\partial \lambda_r} - p\lambda_r^{-1}, \qquad S_{\theta\theta} = \frac{\partial \mathsf{W}}{\partial \lambda_\theta} - p\lambda_\theta^{-1}$$

which are functions of $\lambda(R)$.

4) Solve the equation. Take div $\mathbf{S} = 0$.

$$S_{rr} - S_{\theta\theta} = \frac{\partial \mathsf{W}}{\partial \lambda_r} - \frac{\partial \mathsf{W}}{\partial \lambda_\theta}.$$

We are going to choose λ as a variable. Define $h(\lambda) = \mathsf{W}(\lambda^{-2}, \lambda, \lambda)$, then

$$\frac{\mathrm{d}S_{rr}}{\mathrm{d}\lambda} = -2\frac{S_{rr} - S_{\theta\theta}}{\lambda - \lambda^{-2}} = -\frac{h'(\lambda)}{\lambda^3 - 1},$$

$$\Longrightarrow \boxed{P = \int_{\lambda_b}^{\lambda_a} \frac{h'(\lambda)}{\lambda^3 - 1}}$$

$$\frac{\partial t_r}{\partial r} + \frac{2}{r}(t_r - t_\theta) = 0,$$

$$t_r = \lambda_r W_r - p = \lambda^{-2} W_r - p, \qquad t_\theta = \lambda W_\theta - p, \qquad t_\phi = t_\theta.$$

$$t_r - t_\theta = \lambda^{-2} W_r - \lambda W_\theta$$

$$\Longrightarrow \frac{\partial t_r}{\partial r} + \frac{2}{r}(\lambda^{-2} W_r - \lambda W_\theta) = 0.$$

Introduce auxiliary function, $h(\lambda) = \mathsf{W}(\lambda^{-2}, \lambda, \lambda)$,

$$h'(\lambda) = \frac{\partial h}{\partial \lambda} = W_r.(-2\lambda^{-3}) + W_\theta.1 + W_\phi.1 = -2\lambda^{-1}(\lambda^{-2} W_r - \lambda W_\theta)$$

$$\frac{\partial t_r}{\partial r} = \frac{\partial t_r}{\partial \lambda}\frac{\partial \lambda}{\partial r}$$

$$\lambda = \frac{r}{R(r)}, \qquad \frac{\partial \lambda}{\partial r} = \frac{1}{R} - \frac{rR'}{R^2}$$

$$R^3 = r^3 - a^3 + A^3, \qquad R'R^2 = 3r^3, \qquad R' = \frac{r^2}{R^2} = \lambda^2.$$

$$\frac{\partial \lambda}{\partial r} = \frac{1}{R}(1 - \lambda^3)$$

$$\Longrightarrow \frac{\partial t_r}{\partial r} = \frac{\partial t_r}{\partial \lambda}\frac{1}{R}(1 - \lambda^3) = \frac{\lambda h'(\lambda)}{r}.$$

$$\frac{\partial t_r}{\partial \lambda} = \frac{h'(\lambda)}{1 - \lambda^3}, \qquad \Longrightarrow t_r = \int_{\lambda_a}^{\lambda} \frac{h'(\lambda)}{1 - \lambda^3}\,\mathrm{d}\lambda$$

At $\lambda = \lambda_b$, $t_r = -P$,

$$-P = -\int_{\lambda_a}^{\lambda_b} \frac{h'(\lambda)}{1 - \lambda^3}\,\mathrm{d}\lambda, \qquad \Longrightarrow \qquad P = \int_{\lambda_a}^{\lambda_b} \frac{h'(\lambda)}{1 - \lambda^3}\,\mathrm{d}\lambda = f(\lambda_a).$$

Note

$$\lambda_a = a/A, \qquad \lambda_b = \frac{1}{B}\sqrt[3]{a^3 - A^3 + B^3} = \frac{1}{B}\sqrt[3]{(\lambda_a - 1)A^3 + B^3}.$$

For a given P, we find a, hence the deformation and the value of t_r at all points. Note that $\mathsf{W} = \frac{\mu}{2}(\lambda_r^2 + \lambda_\theta^2 + \lambda_\phi^2)$,

$$\implies h = \frac{\mu}{2}\left(\frac{1}{\lambda^4} + 2\lambda^2\right).$$

Note the nonlinearity in the $1/\lambda^4$ term.

$$\implies \frac{h'}{1-\lambda^3} = -2\mu(\lambda^{-2} + \lambda^{-5}),$$

$$P = -2\mu\left(\frac{1}{\lambda} + \frac{1}{4\lambda^4}\right)\Bigg|_{\lambda_a}^{\lambda_b(\lambda_a)}.$$