LECTURE 9: LINEAR ELASTICITY…CONTINUED

6.5 Incompressible linear elasticity

Recall: Incompressibility:

$$\det \mathbf{F} = 1, \qquad \Longrightarrow \ \det(1+\mathbf{H}) = 1 + \text{tr}\ \ \mathbf{H} + \mathcal{O}(\mathbf{H}^2) = 1$$

Therefore tr $\mathbf{H} = 0 = \text{Div}\ \mathbf{u}$, and

$$\boxed{\text{Div}\ \mathbf{u} = 0} \iff \boxed{\text{tr}\ \ \mathbf{E} = 0}$$

Also

$$\mathbf{T} = -p\mathbb{1} + J^{-1}\mathbf{F}\frac{\partial \mathsf{W}}{\partial \mathbf{F}}, \qquad \sigma = -p\mathbb{1} + C_{ijk\ell}e_{k\ell}$$

For isotropic material,

$$\sigma = 2\mu\mathbf{E} + \lambda(\text{tr}\ \mathbf{E})\mathbb{1} - p\mathbb{1}$$

but

$$\mu = \frac{E}{2(1+\nu)} = \frac{E}{3}.$$

Therefore

$$\boxed{\rho\ddot{\mathbf{u}} = \mathbf{b} - \text{Grad}\ \ p + \mu\Delta\mathbf{u}}$$

and

$$\boxed{\mu = \frac{E}{3}}$$

N.B. Boundary conditions

$$\begin{aligned} \mathbf{u} &= \mathbf{u}^*(t), && \text{on } \partial_1\mathcal{B} \ \ \text{displacements} \\ \tau\mathbf{n} &= \mathbf{t}^*(t), && \text{on } \partial_2\mathcal{B} \ \ \text{traction} \end{aligned}$$

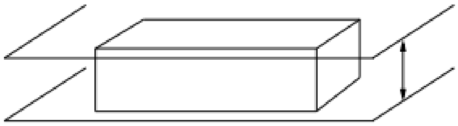
6.5.1 General principles

1) Linear superposition
2) Stresses, strains and displacements are proportional to the loads (or displacements) applied to the solid.
3) If $\partial_2\mathcal{B} = \emptyset$ then there exists one unique solution, only displacements.
4) If only traction and tractions are in equilibrium, then stresses and strains are unique. For initial conditions, there exists one unique $u(t)$.

Some nomenclature about loading

1) Plain strain

$$\mathbf{u} = (u(X,Y), v(X,Y), 0) \implies e_{13} = e_{23} = e_{33} = 0, \qquad \tau_{13} = \tau_{23} = \tau_{31} = \tau_{32} = 0.$$



2) Plane stress

$$\tau_{13} = \tau_{23} = \tau_{33} = 0, \qquad \tau = \begin{bmatrix} \mathbf{a}st & \mathbf{a}st & 0 \\ \mathbf{a}st & \mathbf{a}st & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3) Antiplane strain

$$\mathbf{u} = (0, 0, w(X, Y))$$

4) Pure torsion

$$\mathbf{u} = (-\Omega YZ, \Omega XZ, \Omega\varphi(X, Y))$$

(see problem sheet 6)

6.6 Plane/Strain/Stress Solutions

6.6.1 Plane solutions

(stress or strains)

In cartesian coordinates,

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\tau_{kk}\delta_{ij}$$

$$\frac{\partial \tau_{ij}}{\partial x_i} + b_j = 0$$

Assume that b derives from a potential,

$$b_i = \frac{\partial V}{\partial X_i}, \quad i = 1, 2, \quad b_3 = 0.$$

Plane stresses or strain $\tau_{13} = \tau_{23} = 0$.

6.6.2 Idea

Let

$$\tau_{11} = \frac{\partial \phi}{\partial X_2} - V, \qquad \tau_{22} = \frac{\partial \phi}{\partial X_1} - V, \qquad \tau_{12} = -\frac{\partial^2 \phi}{\partial X_1 \partial X_2}, \qquad \tau_{33} = \beta\nu(\tau_{11} + \tau_{22}),$$

$\beta = 0$ is plane stress and $\beta = 1$ is plane strain.

6.6.3 Equations

$$\frac{\partial \tau_{11}}{\partial X_1} + \frac{\partial \tau_{12}}{\partial X_2} + b_1 = 0, \qquad \frac{\partial \tau_{12}}{\partial X_1} + \frac{\partial \tau_{22}}{\partial X_2} + b_2 = 0.$$

Therefore

$$\frac{\partial}{\partial X_1}\left(\frac{\partial^2 \phi}{\partial X_2^2} - \cancel{V}\right) + \frac{\partial}{\partial X_2}\left(-\frac{\partial^2 \phi}{\partial X_1 \partial X_2}\right) + \cancel{b_1} = 0,$$
$$\frac{\partial}{\partial X_1}\left(-\frac{\partial^2 \phi}{\partial X_1 \partial X_2}\right) + \frac{\partial}{\partial X_2}\left(\frac{\partial^2 \phi}{\partial X_2^2} - \cancel{V}\right) + \cancel{b_2} = 0,$$

and the equations of motion are satisfied. But we do not have an equation for ϕ. We have equations for τ_{ij} or e_{ij}, that is, 6 fields but u_i is 3 components.

6.6.4 Compatibility conditions

Recall: conditions for $\mathbf{F}$: Curl $\mathbf{F} = 0$. For

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Compatibility conditions:

$$\text{Curl Curl } \mathbf{E} = 0,$$

$$\iff \epsilon_{ipm}\epsilon_{jqn}\frac{\partial^2 e_{mn}}{\partial X_p \partial X_q} = 0$$

$$\iff \frac{\partial^2 e_{ij}}{\partial X_k \partial X_\ell} + \frac{\partial^2 e_{k\ell}}{\partial X_i \partial X_j} - \frac{\partial^2 e_{i\ell}}{\partial X_j \partial X_k} - \frac{\partial^2 e_{jk}}{\partial X_i \partial X_\ell} = 0$$

These are $\underline{6}$ relations (but only 3 are independent). For planar problems: $e_{13} = e_{23} = 0$, $\partial e_{ij}/\partial X_3 = 0$,

$$\implies \frac{\partial^2 e_{11}}{\partial X_2^2} + \frac{\partial^2 e_{22}}{\partial X_1^2} - 2\frac{\partial^2 e_{12}}{\partial X_1 \partial X_2} = 0 \qquad (\mathbf{a}st)$$

Now for plane stress we have $\tau_{33} = 0$ and from plane strain we have $\tau_{33} = \nu(\tau_{11} + \tau_{22})$,

$$\iff \tau_{33} = \beta\nu(\tau_{11} + \tau_{22}),$$

which implies

$$e_{11} = \frac{1+\nu}{E}\tau_{11} - \frac{\nu}{E}(1+\beta\nu)(\tau_{11} + \tau_{22})$$
$$e_{22} = \frac{1+\nu}{E}\tau_{22} - \frac{\nu}{E}(1+\beta\nu)(\tau_{11} + \tau_{22})$$
$$e_{12} = \frac{1+\nu}{E}\tau_{12}$$

Insert these into (**a**st) and use $\tau_{11} = \frac{\partial^2\phi}{\partial X_1^2} - V$,

$$\Longrightarrow \frac{\partial^4\phi}{\partial x_1^4} + 2\frac{\partial^4\phi}{\partial x_1^2\partial_2^2} + \frac{\partial^4\phi}{\partial x_2^4} = \frac{1-\beta\nu^2}{1-\nu-2\beta\nu^2}\left(\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2}\right)$$

$$\Longleftrightarrow \boxed{\nabla^4\phi = C_\nu \Delta V}, \qquad C_\nu = \frac{1-\beta\nu^2}{1-\nu-2\beta\nu^2}.$$

Here ∇^4 is the *biharmonic operator* and ϕ is the *Airy potential.* If $\beta = 0$, we have plane stress and $\beta = 1$ is plane strain.

6.6.5 Application

6.7 Elasto-dynamics

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\text{Grad Div }\mathbf{u} = \rho\ddot{\mathbf{u}} \qquad (\mathbf{a}st)$$

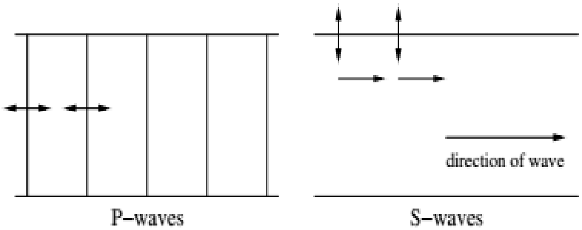
6.7.1 Planar waves

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{a}\sin(\mathbf{k}\cdot\mathbf{x} - ct)$$

Here **a** is the amplitude, **k** is the direction and c is the velocity. We normalize such that $|\mathbf{k}| = 1$.

2 interesting cases:

$\mathbf{a} \parallel \mathbf{k}$ longitudinal – primary, pressure, P-waves.
$\mathbf{a} \perp \mathbf{k}$ transverse – shear, secondary, S-waves.



Let $\varphi(\mathbf{x}, t) = \mathbf{k}\cdot\mathbf{x} - ct$.

$$\text{Note that Div }\mathbf{u} = \mathbf{a}\cdot\mathbf{k}\cos\varphi$$
$$\text{Curl }\mathbf{u} = -\mathbf{a}\times\mathbf{k}\cos\varphi$$

Div $\mathbf{u} = 0$ is transverse, Curl $\mathbf{u} = 0$ is longitudinal.

Substitute $\mathbf{u} = \mathbf{a}\sin\varphi$ in (**a**st). Then

$$\Delta\mathbf{u} = -\mathbf{a}\sin\varphi$$
$$\text{Grad Div }\mathbf{u} = \text{Grad}(\mathbf{a}\cdot\mathbf{k}\cos\varphi) = (\mathbf{a}\cdot\mathbf{k})\mathbf{k}(-\sin\varphi)$$

Therefore $\ddot{\mathbf{u}} = -c^2\mathbf{a}\sin\varphi$ and

$$\mu\mathbf{a} + (\lambda+\mu)(\mathbf{a}\cdot\mathbf{k})\mathbf{k} = \rho c^2\mathbf{a}$$

This is a linear operator on $\mathbf{a}$. Define A the *acoustic tensor*,

$$\mathsf{A} = \frac{1}{\rho}\left(\mu 1 + (\lambda+\mu)\mathbf{k}\otimes \mathrm{k}\right) [\mathsf{A}]_{ij} = \frac{1}{\rho}\left(\mu\delta_{ij} + (\lambda+\mu)k_i k_j\right)$$

so that we have the eigenvalue problem

$$\mathsf{A}\mathsf{a} = c^2\mathbf{a}$$

1) $\mathbf{a} = \alpha\mathbf{k}$

$$\alpha A_{ij}k_j = \frac{1}{\rho}(\mu k_i + (\lambda+\mu)\underbrace{k_j k_j}_{1} k_i) = c^2\alpha k_i$$

$$\Longrightarrow \frac{\lambda+2\mu}{\rho} = c^2, \qquad c_L = \sqrt{\frac{\lambda+2\mu}{\rho}}$$

2) $\mathbf{a}\perp\mathbf{k}$, $a_i k_i = 0$.

$$A_{ij}a_j = \frac{1}{\rho}\left(\mu a_i + (\lambda+\mu)k_i k_j a_j\right) = c^2 a_i$$

$$\Longrightarrow c^2 = \frac{\mu}{\rho}, \qquad c_T = \sqrt{\frac{\mu}{\rho}} < \sqrt{\frac{\lambda+2\mu}{\rho}},$$

i.e. slower than c_L.
Note also

$$c_L = \sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}},$$

where $1-2\nu = 0$ for an incompressible material. Therefore $c_L \to \infty$ as $\nu \to 1/2$.
Also note

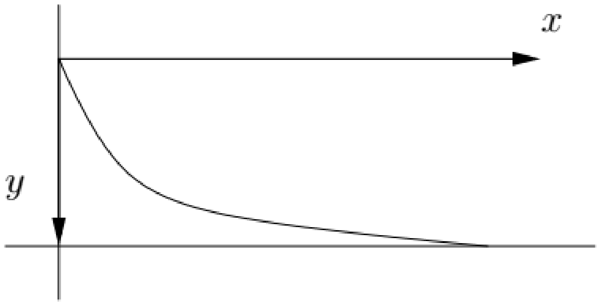
$$c_T^2 = \mu/\rho \quad \Longrightarrow \quad \mu = c_T^2\rho$$
$$c_L^2 = \frac{\lambda}{\rho} + 2c_T^2, \quad \Longrightarrow \quad \rho c_L^2 - 2\rho c_T^2$$

$$\Longrightarrow \boxed{\ddot{\mathbf{u}} = c_T^2\Delta\mathbf{u} + (c_L^2 - c_T^2)\text{Grad Div } \mathbf{u}} \qquad (\mathbf{a}st\mathbf{a}st)$$

6.8 Rayleigh waves

$$\mathbf{u} = \Re\begin{bmatrix} Ae^{-by}\exp(ik(x-ct)) \\ Be^{-by}\exp(ik(x-ct)) \\ 0 \end{bmatrix}$$

$A, B \in \mathbb{C}$, $b \in \mathbb{R}_0^+$.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