C6.1a Solid Mechanics. Formulas.

A few key formulas and definitions that we are using throughout this course.

Tensor calculus Here ϕ, $\mathbf{v}$, $\mathbf{T}$ are, respectively, scalar, vector and 2^{nd}-order tensor fields defined on a moving body. Upper case refer to the reference configuration, lower case to the current configuration.

$\mathbf{F} = \operatorname{Grad}\mathbf{x} = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j$	Deformation Gradient	$(T1)$
$J = \det\mathbf{F}$	Determinant of $\mathbf{F}$	$(T2)$
$\operatorname{grad}\mathbf{v} = \frac{\partial \mathbf{v}}{\partial x_i} \otimes \mathbf{e}_i$	Definition of the gradient of a vector	$(T3)$
$\operatorname{grad}\mathbf{T} = \frac{\partial \mathbf{T}}{\partial x_i} \otimes \mathbf{e}_i$	Definition of the gradient of a tensor	$(T4)$
$\operatorname{div}\mathbf{T} = \frac{\partial T_{ij}}{\partial x_i}\mathbf{e}_j$	Definition of the divergence of a tensor	$(T5)$
$\operatorname{Grad}\phi = \mathbf{F}^{\mathbf{T}}\operatorname{grad}\phi$	Gradients of a scalar	$(T6)$
$\operatorname{Grad}\mathbf{v} = (\operatorname{grad}\mathbf{v})\mathbf{F}$	Gradients of a vector	$(T7)$
$\operatorname{Div}\mathbf{v} = J\operatorname{div}(J^{-1}\mathbf{F}\mathbf{v})$	Divergences of a vector	$(T8)$
$\operatorname{Div}\mathbf{T} = J\operatorname{div}(J^{-1}\mathbf{F}\mathbf{T})$	Divergences of a tensor	$(T9)$
$\operatorname{div}(J^{-1}\mathbf{F}) = 0$	An important identity	$(T10)$
$\frac{\partial}{\partial\lambda}(\det\mathbf{T}) = (\det\mathbf{T})\operatorname{tr}\left(\mathbf{T}^{-1}\frac{\partial\mathbf{T}}{\partial\lambda}\right)$	A useful identity. λ is a scalar	$(T11)$

Kinematics

$\mathbf{F} = \operatorname{Grad}\ \mathbf{x}(\mathbf{X}, t)$	The deformation gradient	$(K1)$
$J = \det\mathbf{F}$	Determinant of $\mathbf{F}$	$(K2)$
$d\mathbf{x} = \mathbf{F}d\mathbf{X}$	Transformation of line element	$(K3)$
$d\mathbf{a} = J\mathbf{F}^{\text{-T}}d\mathbf{A}$	Transformation of area element	$(K4)$
$dv = JdV$	Transformation of volume element	$(K5)$
$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$	Right Cauchy-Green tensor	$(K6)$
$\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$	Left Cauchy-Green tensor	$(K7)$
$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{1}\right)$	Euler strain tensor	$(K8)$
$\mathbf{L} = \operatorname{grad}\ \mathbf{v}$	Velocity gradient	$(K9)$
$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$	Evolution of the deformation gradient($\mathbf{v}$: velocity)	$(K10)$
$\dot{J} = J\operatorname{div}\ \mathbf{v}$	Evolution of the volume element	$(K11)$
$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^{\mathrm{T}}\right)$	Eulerian strain rate tensor	$(K12)$
$\mathbf{W} = \frac{1}{2}\left((\mathbf{L} - \mathbf{L}^{\mathrm{T}}\right)$	Rate of rotation tensor	$(K13)$

List of assumptions

- **Continuum assumption.** We consider a body with reference configuration $\mathcal{B}_0 \subset \mathbb{R}^3$. At time t, the body occupies the current configuration $\mathcal{B}_t \subset \mathbb{R}^3$. A material point, initially at $\mathbf{X} \in \mathcal{B}_0$ is mapped to a point $\mathbf{x} \in \mathcal{B}_t$ by the one-parameter mapping $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ so that $\boldsymbol{\chi} : \mathcal{B}_0 \to \mathcal{B}_t$. The continuum assumption states that $\boldsymbol{\chi}$ is a bijection mapping for all time t. This implies that we can write $\mathbf{x} = \boldsymbol{\chi}^{-1}(\mathbf{X})$. We further assume that this mapping is twice continuously differentiable in $\mathbf{X}$ and t. This assumption can be relaxed in problems involving phase boundaries (with possible jumps in the first derivative). In many instances and applications, we will assume that $\boldsymbol{\chi}$ is actually smooth.

- **Conservation of mass.** We assume the existence of a scalar density function $\rho = \rho(\mathbf{x})$ defined on the body $\mathcal{B}_t$ and whose integral over any material subset $\Omega_t \subset \mathcal{B}_t$ of the body remains constant in time. So that $\frac{d}{dt}\int_{\Omega_t} \rho(\mathbf{x}) dv = 0$

- **Balance of linear momentum.** We assume that the rate of change of linear momentum of an arbitrary material subset $\Omega_t \subset \mathcal{B}_t$ is equal to sum of all the forces acting on Ω_t.

- **Balance of angular momentum.** We assume that the rate of change of angular momentum of an arbitrary material subset $\Omega_t \subset \mathcal{B}_t$ with respect to a given point is equal to sum of all the torques acting on Ω_t with respect to the same point.

- **Polar media.** For polar media, we assume that the body is not subject to body or contact torques.

- **Cauchy's postulate media.**