LECTURE 7: DIFFERENTION AND THE DIFF COMMAND

The Symbolic Math Toolbox can find the derivatives of expressions of one or several variables. In this section we introduce the `diff` command, which is the basic differentiation operator for expressions, and illustrate its use in different situations including finding the coordinates of stationary points.

We start with an example; the derivative of $\sin x$ with respect to x is found with the command

```
>> syms x
>> diff(sin(x), x)
```

The general syntax for differentiating an expression `expr` with respect to a variable `var` is

```
>> diff(expr, var);
```

and the derivative of $e^{x^2+a^2}$ with respect to x is found with the commands

```
>> syms x a
>> z = exp(x^2 + a^2);
>> d1 = diff(z, x)
```

The second derivative of $e^{x^2+a^2}$ with respect to x is

```
>> diff(d1, x)
```

and this same result can also be found all in one go by typing

```
>> diff(z, x, 2)
```

So the fifth derivative of $\sin(x^2)$ is achieved

```
>> diff(sin(x^2), x, 5)
```

Partial differentiation[3] of expressions of more than one variable is performed by a straightforward extension of this procedure. For example, consider the expression $\sin x \cos y$; then its second derivatives are found as follows. First define $z = \sin x \cos y$:

```
>> syms x y
>> z = sin(x)*cos(y)
```

Then the second derivative $\partial^2 z/\partial x^2$ is given by

```
>> diff(z, x, 2)
```

and $\partial^2 z/\partial y^2$ is found with

```
>> diff(z, y, 2)
```

or by nesting the `diff` command

```
>> diff(diff(z, y), y)
```

Thus one way to evaluate the mixed second derivative $\frac{\partial^2 z}{\partial x \partial y}$ is:

```
>> diff(diff(z, x), y)
```

This is a bit cumbersome, particularly if you want something like $\frac{\partial^5 z}{\partial x^2 \partial y^3}$. In MATLAB R2013b and later versions, you can do:

```
>> diff(z, x, x, y, y, y)
```

(although this won't work with earlier versions.) For example to verify that the order of differentiation does not matter here:

```
>> diff(z, y, x)
>> diff(z, x, y)
```

Sometimes it is necessary to find the derivative of an expression at a particular point, rather than as a function of the independent variable(s). This is achieved with a combination of the `diff`, `subs` and sometimes `double` commands. Thus the value of the first derivative of $\sin x$ at $x = 6$ is given by the commands

```
>> f = diff(sin(x), x);
>> subs(f, x, 6)
```

and to get a decimal approximation:

```
>> double(subs(f, x, 6))
```

The following exercises give practice in the `diff` command and also revise some earlier commands such as `solve` and `plot`.

Exercise 2.7 Find

$$\frac{d}{dx}\left(\frac{(x^2+1)^4}{e^{2x}}\right)$$

and evaluate it at $x = 1$. □

Exercise 2.8 Plot the second derivative of $x^4/(1+x^2)$ for $-5 \le x \le 5$. □

Exercise 2.9 Given $y = 12x^5 - 15x^4 + 20x^3 - 330x^2 + 600x + 2$, find the (x, y) coordinates of any stationary points (that is, those for which $y'(x) = 0$). Use MATLAB to plot y so that the real stationary points are visible. Can you also make a plot to show the complex stationary points? □

Exercise 2.10 If $z = \ln(x^2 + y^2)$ find the value of the constant a such that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = ae^{-z}.$$

□

Exercise 2.11 Execute this code:

```
>> diff(sym(5))
>> diff(5)
```

and explain the results (hint: what does `diff([1 2 5 9])` do?). □

2.3.1 Differentiation of unknown functions

So far only the derivatives of expressions with an explicit form (in terms of one or more independent variables) have been found. In this section we look at how to differentiate arbitrary functions, say $y(x)$ or $f(x, y)$, without first giving them an explicit form. For example, we might want to find dy/dx (as a function of y and x) given that $x^2 + y^2 = 3$. However, typing

```
>> syms x y
>> diff(y, x)
```

returns `0` because MATLAB does not know that we intended `y` to depend on `x`; `x` and `y` are free independent variables and are treated as such.

This difficulty is addressed in the Symbolic Math Toolbox by using `y(x)`; MATLAB interprets this syntax to mean that `y(x)` depends on `x`. Thus we have

```
>> clear
>> syms y(x)
>> diff(y,x)
>> diff(y,x,2)
```

Note here the `syms y(x)` also makes `x` a symbolic variable. In fact let's take a closer look

```
>> whos
```

and see that `y(x)` is a "`symfun`"—a symbolic function, and `x` is a `sym`. In fact, the command "`syms y(x)`" is a shortcut for:

```
>> x = sym('x')
>> y(x) = sym('y(x)')
```

Differentiation and the "symfun"

The syntax here is quite important. Note the following:

```
>> x = sym('x')
>> f = sym('f(x)')    % this is probably a mistake
>> g(x) = sym('g(x)')
```

Here g is a symfun but f is just the expression $f(x)$: probably not what was intended. Note also the difference

```
>> A = diff(g, x)        % yes, A is a symfun for the deriv
>> B = diff(g(x), x)     % gives an expression, not a symfun
>> C = diff(f, x)        % gives an expression, not a symfun
>> %D = diff(f(x), x)    % would give an error
>> whos A B C
```

The distinction between an expression and a symfun is usually not important but in some cases—such as dealing with differential equations—it makes an important difference. In summary, when it matters, the form `syms f(x)` is probably less error-prone.

Functions of multiple variables can also be manipulated using symfuns

```
>> syms f(x,y)
>> f
>> diff(f, x)
>> diff(f, x, 2)
>> diff(f, x, y)
>> diff(f, x, x, y, y, y)
```

In versions of MATLAB before R2013b, the last two would need be nested as in

```
>> diff(diff(f, x), y)
>> diff(diff(f, x, 2), y, 3)
```

Using symfuns, the diff command knows the usual rules of differentiation, for instance the addition, multiplication and quotient rules:

```
>> syms f(x) g(x)
>> diff(f + g, x)
>> diff(f * g, x)
>> diff(f / g, x)
```

To illustrate an application of this theory, suppose that we wish to find dy/dx (in terms of x and y) given that $x^2 + y^2 = 3$. To do this with MATLAB, first define the given explicit equation by typing

```
>> clear
>> syms y(x)
>> eq1 = x^2 + y(x)^2 == 3
```

and then differentiate this equation, assigning the output equation to a new name

```
>> eq2 = diff(eq1, x)
```

Let's ask MATLAB to solve the resulting expression for dy/dx:

```
>> solve(eq2, diff(y, x))      % fail
>> solve(eq2, diff(y(x), x))     % fail
```

Hmmm, we'll have to try a bit harder:

```
>> % use subs to replace the deriv with "dydx"
>> syms dydx
>> eq3 = subs(eq2, diff(y(x), x), dydx)
>> % now solve for dydx
>> solve(eq3, dydx)
```

Exercise 2.12 Find the derivatives of

$$\sin f(x), \quad \sin\left(e^{f(x)}\right), \quad \exp\left(\sqrt{1+f(x)g(x)}\right).$$

□

Exercise 2.13 Find dz/dp implicitly (in terms of z and p) given that

$$p^3 + z(p)^2 + 3pz(p) = 0.$$

Try to convince MATLAB to isolate dz/dp, that is, solve for dz/dp. □

2.4 Evaluating limits

This short section introduces the command `limit`. Its syntax is fairly self-explanatory; for example to find

$$\lim_{x\to 0} \frac{\sin x}{x}$$

type

```
>> syms x
>> limit(sin(x)/x, x, 0)
```

Left and right limits can also be found; for example to find

$$\lim_{x\to 3^+} \frac{x^2-4}{x^2-5x+6}$$

use the command

```
>> syms x
>> expr = (x^2-4)/(x^2-5*x+6);
>> limit(expr, x, 3, 'right')
```

Exercise 2.14 Use MATLAB to evaluate the following limits

$$\text{(i)}\quad \lim_{x\to 0} \frac{\tan x - x}{x - \sin x}, \qquad \text{(ii)}\quad \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x.$$

In each case, make a plot that clearly demonstrates that the limits are correct. □