LECTURE 4:

TWO-DIMENSIONAL INCOMPRESSIBLE IRROTATIONAL FLOW

Velocity potential and streamfunction

We now focus on purely two-dimensional flows, in which the velocity takes the form

$$\boldsymbol{u}(x,y,t) = u(x,y,t)\boldsymbol{i} + v(x,y,t)\boldsymbol{j}. \tag{2.1}$$

With the velocity given by (2.1), the vorticity takes the form

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\boldsymbol{k}. \tag{2.2}$$

We assume throughout that the flow is *irrotational*, *i.e.* that $\boldsymbol{\nabla} \times \boldsymbol{u} \equiv \boldsymbol{0}$ and hence

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \tag{2.3}$$

We have already shown in Section 1 that this condition implies the existence of a *velocity potential* ϕ such that $\boldsymbol{u} \equiv \boldsymbol{\nabla}\phi$, that is

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}. \tag{2.4}$$

We also recall the definition of ϕ as

$$\phi(x,y,t) = \phi_0(t) + \int_{\boldsymbol{0}}^{\boldsymbol{x}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = \phi_0(t) + \int_{\boldsymbol{0}}^{\boldsymbol{x}} (u\,\mathrm{d}x + v\,\mathrm{d}y), \tag{2.5}$$

where the scalar function $\phi_0(t)$ is arbitrary, and the value of $\phi(x,y,t)$ is independent of the integration path chosen to join the origin $\boldsymbol{0}$ to the point $\boldsymbol{x} = (x,y)$. This fact is even easier to establish when we restrict our attention to two dimensions. If we consider two alternative paths, whose union forms a simple closed contour C in the (x,y)-plane, Green's Theorem implies that

$$\oint_C (u\,\mathrm{d}x + v\,\mathrm{d}y) \equiv \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \mathrm{d}x\mathrm{d}y, \tag{2.6}$$

where S is the region enclosed by C, and the right-hand side of (2.6) is zero by virtue of (2.3).

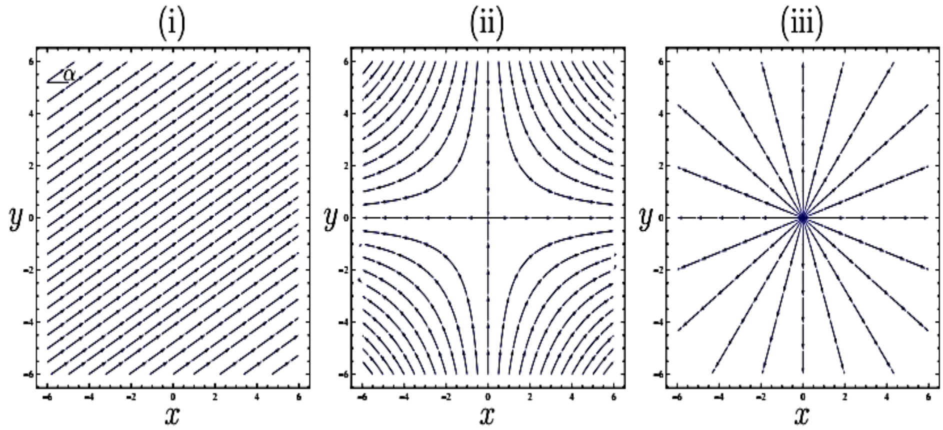

Figure 2.1: Streamline plots for (i) uniform flow; (ii) stagnation point flow; (iii) line source.

We also assume that the flow is *incompressible*, so that $\nabla \cdot \boldsymbol{u} = 0$, which in two dimensions reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}. \tag{2.7}$$

By substituting (2.4) into (2.7), we find that ϕ satisfies Laplace's equation, that is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = 0. \tag{2.8}$$

As noted in Section 1, Laplace's equation is linear, and therefore much easier to solve than the nonlinear Euler equations. We will see that many flows may be constructed using well-known simple solutions of Laplace's equation. Once we have solved for ϕ, the velocity components are easily recovered from (2.4), and the pressure may be found using Bernoulli's Theorem. We recall from Section 1 that this reads

$$\frac{p}{\rho} + \frac{1}{2}|\nabla \phi|^2 + \chi \text{ is constant in steady irrotational flow,} \tag{2.9}$$

or

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}|\nabla \phi|^2 + \chi = F(t) \text{ in unsteady irrotational flow.} \tag{2.10}$$

Example 2.1 ***Uniform flow***
A linear function of x and y trivially satisfies Laplace's equation. The velocity potential

$$\phi = Ux\cos\alpha + Uy\sin\alpha, \tag{2.11}$$

where U and α are constants, represents the velocity field

$$(u, v) = U(\cos\alpha, \sin\alpha). \tag{2.12}$$

This corresponds to uniform flow, *with speed U, in a direction making an angle α with the x-axis, as shown in* Figure 2.1(i).

Example 2.2 *Stagnation point flow*
The velocity potential

$$\phi = \frac{1}{2}\left(x^2 - y^2\right) \tag{2.13}$$

clearly satisfies Laplace's equation and corresponds to the stagnation point flow

$$(u, v) = (x, -y) \tag{2.14}$$

considered previously in Section 1. *We recall that the streamlines are hyperbolae, as illustrated in* Figure 2.1(ii).

Example 2.3 *Line source*
We recall that the two-dimensional Laplace's equation may be written as

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0, \tag{2.15}$$

in terms of plane polar coordinates (r, θ). *If we seek a radially-symmetric solution, with* $\phi = \phi(r)$, *we find that*

$$\phi = \frac{Q}{2\pi}\log r, \tag{2.16}$$

where Q *is a constant of integration, and the additional arbitrary constant may be set to zero without loss of generality. The corresponding velocity field is given by*

$$\boldsymbol{u} = \frac{\mathrm{d}\phi}{\mathrm{d}r}\boldsymbol{e}_r = \frac{Q}{2\pi r}\boldsymbol{e}_r, \tag{2.17}$$

where $\boldsymbol{e}_r$ *is the unit-vector in the r-direction. The velocity (2.17) may be rewritten in terms of Cartesian variables as*

$$\boldsymbol{u} = (u, v) = \frac{Q}{2\pi\left(x^2 + y^2\right)}(x, y). \tag{2.18}$$

We see that the fluid flows radially outward, at a speed that increases without bound as we approach the origin. The streamlines are straight rays pointing outward from the origin, as shown in Figure 2.1(iii). *This flow is called a* line source*: a "source" because fluid is being squirted out of the origin, and a "line" because the point* $(0, 0)$ *in the* (x, y)*-plane really corresponds to a line (the* z*-axis) in three-dimensional space.*

The constant Q *is called the* source strength. *To see why, let us calculate the rate at which fluid crosses a contour* C *containing the origin, namely*

$$\oint_C \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}s,$$

where $\boldsymbol{n}$ *is the outward-pointing unit normal to* C. *The integral is particularly straightforward if we choose* C *to be the circle* $r = a$, *for some constant radius* a, *in which case* $\boldsymbol{n} \equiv \boldsymbol{e}_r$ *and hence*

$$\oint_C \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}s = \int_0^{2\pi} \frac{Q}{2\pi a}\boldsymbol{e}_r\cdot\boldsymbol{e}_r a\,\mathrm{d}\theta = Q. \tag{2.19}$$

It is easily shown that any other simple closed curve C *containing the origin gives the same value of the flux. Therefore* Q *is the rate at which fluid is produced by the source.*

If Q *is negative, then fluid is being consumed rather than produced, and the origin is called a* line sink, *of strength* $|Q|$.

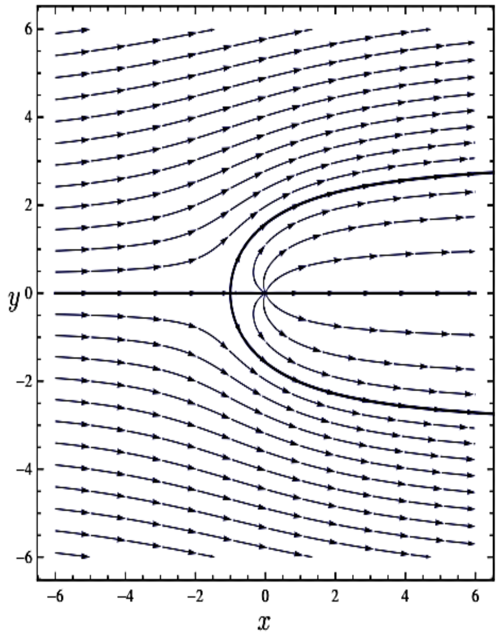

Figure 2.2: Streamlines for the flow produced by a line source of strength Q in a uniform flow at speed U in the x-direction. (In this plot, $Q = 2\pi U$.)

One of the great advantages of dealing with a *linear* partial differential equation is that the result of combining two flows may be found by simply adding the corresponding velocity potentials. This is emphatically *not* the case for the nonlinear Euler equations in general.

Example 2.4 ***Line source in a uniform flow***
Consider the flow produced by placing a line source of strength Q at the origin in a uniform flow of speed U in the x-direction. The velocity potentials due to the source and the uniform flow are Ux and $(Q/2\pi)\log r$ respectively, and the potential of the combined flow is thus

$$\phi = Ux + \frac{Q}{2\pi}\log r. \tag{2.20}$$

The corresponding velocity components are

$$u = U + \frac{Qx}{2\pi\,(x^2+y^2)}, \qquad v = \frac{Qy}{2\pi\,(x^2+y^2)}, \tag{2.21}$$

and the streamlines are shown in Figure 2.2.

It is easily seen that there is just one stagnation point (i.e. point where the velocity is zero) in the flow, at $(x, y) = (-Q/2\pi U, 0)$. The separatrix crossing the x-axis through this point separates the fluid emanating from the source from the fluid carried by the uniform flow.

By analogy with (2.3) and (2.4), we could instead start from (2.7) and deduce the existence of a *streamfunction* $\psi(x, y, t)$ satisfying

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}. \tag{2.22}$$

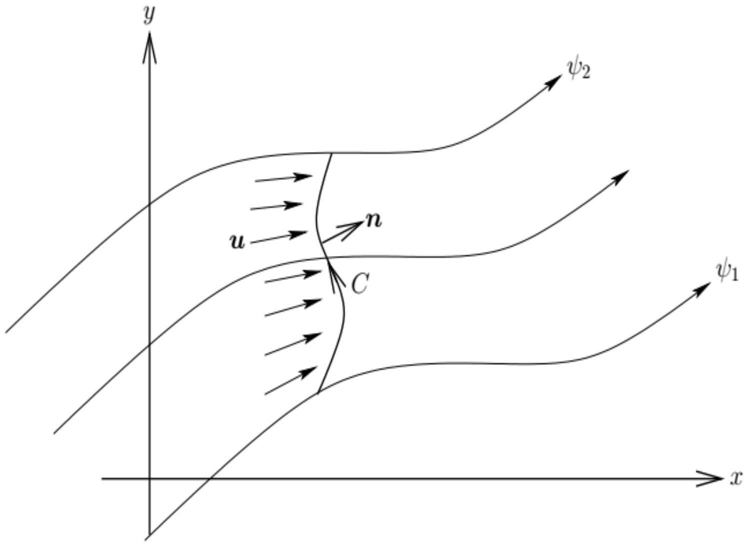

Figure 2.3: Schematic of a curve C joining two streamlines on which $\psi = \psi_1$ and $\psi = \psi_2$.

A handy shorthand for (2.22) is

$$\boldsymbol{u} = \boldsymbol{\nabla} \times (\psi \boldsymbol{k}). \tag{2.23}$$

By comparison with (2.5), we infer that ψ may be defined as

$$\psi(x, y, t) = \psi_0(t) + \int_{\boldsymbol{0}}^{\boldsymbol{x}} (u\,\mathrm{d}y - v\,\mathrm{d}x), \tag{2.24}$$

where the function $\psi_0(t)$ is arbitrary: just as for ϕ, any function of t may be absorbed into ψ without influencing the velocity components given by (2.22). Again, the integral in (2.24) is independent of the path taken from the origin to the point $\boldsymbol{x}$, and the proof follows quickly from Green's Theorem.

The representation (2.22) implies that ψ satisfies

$$\boldsymbol{u} \cdot \boldsymbol{\nabla}\psi = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right) \cdot \left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right) \equiv 0, \tag{2.25}$$

and it follows that *the streamfunction is constant along streamlines.* The contours of ψ therefore give us a handy tool for plotting the streamlines of a given flow.

Now let us consider two neighbouring streamlines, on which ψ takes the constant values $\psi = \psi_1$ and $\psi = \psi_2$, say. The *flux*, *i.e.* the net flow of fluid, between the two streamlines, is given by the integral

$$\text{flux} = \int_C \boldsymbol{u} \cdot \boldsymbol{n}\,\mathrm{d}s, \tag{2.26}$$

where C is a smooth path joining the two streamlines, with unit normal $\boldsymbol{n}$, as shown schematically in Figure 2.3. Substituting for the velocity components from (2.4) and identifying $\boldsymbol{n}\,\mathrm{d}s$ with $(\mathrm{d}y, -\mathrm{d}x)$, we can rewrite (2.26) as

$$\text{flux} = \int_C \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right)\cdot(\mathrm{d}y, -\mathrm{d}x) = \int_C \left(\frac{\partial\psi}{\partial x}\mathrm{d}x + \frac{\partial\psi}{\partial y}\mathrm{d}y\right) = [\psi]_C = \psi_2 - \psi_1, \quad (2.27)$$

where the notation $[\psi]_C$ respresents the change in the value of ψ from one end of C to the other. Therefore,

the change in the value of ψ from one streamline to another is equal to the net flux of fluid between them. (2.28)

By substituting (2.22) into (2.3), we find that ψ, like ϕ, satisfies the two-dimensional Laplace's equation:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \nabla^2\psi = 0. \quad (2.29)$$

Again, we can pose familiar solutions of Laplace's equation for ψ and read off the corresponding velocity fields from (2.22). We begin by showing how the flows considered in Examples 2.1–2.4 may be described using the streamfunction.

Example 2.5 ***Uniform flow***
The streamfunction corresponding to the uniform flow (2.12) *is easily seen to be*

$$\psi = Uy\cos\alpha - Ux\sin\alpha, \quad (2.30)$$

up to an arbitrary constant. Hence the streamlines are given by $\psi =$ *constant, that is,*

$$y = x\tan\alpha + \textit{constant}, \quad (2.31)$$

in agreement with Figure 2.1(i).

Example 2.6 ***Stagnation point flow***
The streamfunction corresponding to the stagnation point flow (2.14), *satisfies the differential equations*

$$\frac{\partial\psi}{\partial x} = y, \qquad \frac{\partial\psi}{\partial y} = x. \quad (2.32)$$

We easily deduce that

$$\psi = xy \quad (2.33)$$

up to an arbitrary constant. The streamlines plotted in Figure 2.1(ii) *are the contours of* ψ, *namely the hyperbolae* $xy =$ *constant.*

The representation (2.23) is helpful if we wish to use the streamfunction in plane polar coordinates (r, θ). Using the formulae given in §2.5.2, we find that

$$\boldsymbol{\nabla}\times(\psi\boldsymbol{k}) \equiv \frac{1}{r}\frac{\partial\psi}{\partial\theta}\boldsymbol{e}_r - \frac{\partial\psi}{\partial r}\boldsymbol{e}_\theta, \quad (2.34)$$

and the polar velocity components u_r and u_θ are therefore related to ψ by

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = -\frac{\partial\psi}{\partial r}. \quad (2.35)$$

Example 2.7 ***Line source***

By substituting the velocity field (2.17) *due to a line source into equation* (2.35), *we find that the corresponding streamfunction must satisfy*

$$\frac{\partial \psi}{\partial r} = 0, \qquad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{Q}{2\pi r}. \tag{2.36}$$

Hence, up to the addition of an arbitrary constant,

$$\psi = \frac{Q}{2\pi}\theta. \tag{2.37}$$

The contours of ψ *are the rays* $\theta =$ *constant, in agreement with the streamlines plotted in* Figure 2.1(iii).

Example 2.7 demonstrates that the streamfunction due to a point source is *multivalued.* We should not be too surprised by this, since our uniqueness proof relied on Green's Theorem, which does not apply to a velocity field like (2.18), which is not differentiable (or even bounded) at the origin. Furthermore, the jump of Q in ψ as θ increases from 0 to 2π corresponds to the *flux* between these two streamlines, according to (2.28). This agrees with the result that Q is the rate at which the source produces fluid.

Example 2.8 ***Line source in a uniform flow***

The streamfunction corresponding to the flow (2.21) *may be found by combining the streamfunctions* (2.30) *and* (2.37) *found above, to obtain*

$$\psi = Uy + \frac{Q}{2\pi}\theta. \tag{2.38}$$

The streamlines plotted in Figure 2.2 *are the contours of this function, which may be written in the form*

$$r = \left(\frac{Q}{2\pi U}\right)\frac{C-\theta}{\sin\theta}, \tag{2.39}$$

where different streamlines correspond to different choices of the constant C.

As our final example, we examine the implications of a radially-symmetric streamfunction, by analogy with (2.3).

Example 2.9 ***Line vortex***

If the streamfunction depends only upon the radial coordinate r, *then, as in equation* (2.16), *it must take the form*

$$\psi = \frac{-\Gamma}{2\pi}\log r, \tag{2.40}$$

up to the addition of an arbitrary constant. The minus sign in front of the integration constant Γ *is included for later convenience. We see that this will give circular streamlines* $r =$ *constant, as shown in* Figure 2.4. *By using (2.35), we find that the velocity field is given by*

$$\boldsymbol{u} = -\frac{\mathrm{d}\psi}{\mathrm{d}r}\boldsymbol{e}_\theta = \frac{\Gamma}{2\pi r}\boldsymbol{e}_\theta. \tag{2.41}$$

Hence the fluid rotates around the origin, with a speed that increases without bound as $r \to 0$. *This flow is called a* line vortex, *the line in question being the* z*-axis in three-dimensional space.*

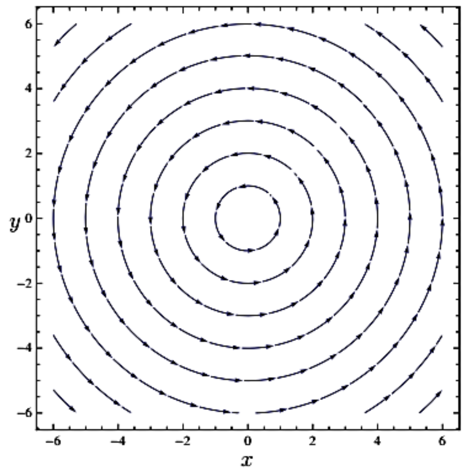

Figure 2.4: Streamlines due to a vortex at the origin.

The constant Γ *is called the* vortex strength. *We recall that the strength of a source measures the rate at which it produces fluid. In contrast, we will see that the strength of a vortex corresponds to the* circulation *around a contour* C *containing the origin, namely*

$$\oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} \equiv \oint_C (u\,\mathrm{d}x + v\,\mathrm{d}y).$$

We make the integral as simple as possible by choosing C *to be the circle* $r = a$, *for some constant radius* a, *in which case* $(\mathrm{d}x, \mathrm{d}y) \equiv \boldsymbol{e}_\theta a\,\mathrm{d}\theta$ *and hence*

$$\oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = \int_0^{2\pi} \frac{\Gamma}{2\pi a} \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta\, a\,\mathrm{d}\theta = \Gamma. \tag{2.42}$$

It is easily shown that the circulation around any other simple closed curve C *containing the origin is also equal to* Γ.

If Γ *is positive, the vortex rotates in an anticlockwise sense, as in* Figure 2.4*; a negative value of the circulation corresponds to clockwise rotation.*

We recall that the circulation is supposed to be identically zero in an irrotational flow. However, this result relies on Green's Theorem, which does not apply to a singular velocity field such as (2.41). A vortex may be thought of as a *source of vorticity* in an otherwise irrotational flow.