LECTURE 7: KUTTA–JOUKOWSKI LIFT THEOREM

Consider steady uniform flow at speed U past an obstacle B, where there is a circulation Γ about B but there are no singularities in the flow. Then the obstacle experiences a drag force *D parallel to the flow and* lift force *L perpendicular to the flow given by*

$$D = 0, \qquad L = -\rho U \Gamma. \tag{2.94}$$

Proof Since there are assumed to be no singularities in the flow, $\mathrm{d}w/\mathrm{d}z$ must be holomorphic outside B and must therefore have a Laurent expansion of the form

$$\frac{\mathrm{d}w}{\mathrm{d}z} \sim U\mathrm{e}^{-\mathrm{i}\alpha} + \frac{b_1}{z} + \mathrm{O}\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty. \tag{2.95}$$

The first term in (2.95) corresponds to the imposed uniform flow at infinity, which we allow to approach at an arbitrary inclination α to the x-axis.

We recall from Blasius' Theorem that the components of the force on B are given by (2.85). Since there are no singularities in the flow, we can use Cauchy's Theorem to deform the contour of integration from ∂B to a large circle C of radius R, as shown schematically in Figure 2.17(i):

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2} \oint_{\partial B} \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z = \frac{\mathrm{i}\rho}{2} \oint_{C} \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z. \tag{2.96}$$

We parametrise C using $z = R\mathrm{e}^{\mathrm{i}\theta}$ and use (2.95) to obtain

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2} \int_0^{2\pi} \left\{U\mathrm{e}^{-\mathrm{i}\alpha} + \frac{b_1\mathrm{e}^{-\mathrm{i}\theta}}{R} + \mathrm{O}\left(\frac{1}{R^2}\right)\right\}^2 \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta, \tag{2.97}$$

and, letting $R \to \infty$, we find that

$$F_x - \mathrm{i}F_y = -2\pi\rho b_1 U\mathrm{e}^{-\mathrm{i}\alpha}. \tag{2.98}$$

The drag and lift forces are defined to be the components parallel and perpendicular to the flow, as shown in Figure 2.17(ii). Simple trigonometry tells as that they are related to F_x and F_y by

$$D + \mathrm{i}L = (F_x + \mathrm{i}F_y)\,\mathrm{e}^{-\mathrm{i}\alpha}, \tag{2.99}$$

and we deduce from (2.98) that

$$D - \mathrm{i}L = -2\pi\rho b_1 U. \tag{2.100}$$

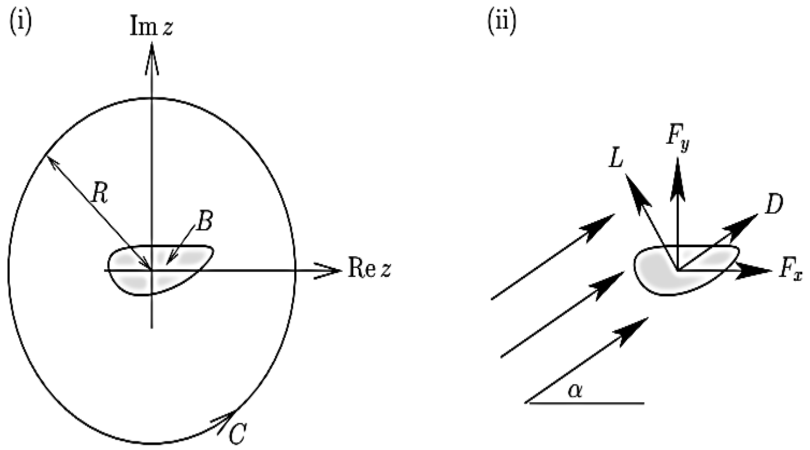

Figure 2.17: (i) Schematic showing deformation of the boundary ∂B of an obstacle B being deformed to a large circular contour of radius R. (ii) The relations between the drag and lift forces and the components F_x and F_y.

To evaluate the constant b_1, we note that

$$\oint_C \frac{\mathrm{d}w}{\mathrm{d}z}\,\mathrm{d}z = \int_0^{2\pi} \left\{ U\mathrm{e}^{-\mathrm{i}\alpha} + \frac{b_1 \mathrm{e}^{-\mathrm{i}\theta}}{R} + \mathrm{O}\left(\frac{1}{R^2}\right) \right\} \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = 2\pi \mathrm{i} b_1. \tag{2.101}$$

On the other hand, the real and imaginary parts of the left-hand side give us

$$\oint_C \frac{\mathrm{d}w}{\mathrm{d}z}\,\mathrm{d}z = \oint_C (u - \mathrm{i}v)(\mathrm{d}x + \mathrm{i}\mathrm{d}y) = \oint_C (u\,\mathrm{d}x + v\,\mathrm{d}y) + \mathrm{i}\oint_C (u\,\mathrm{d}y - v\,\mathrm{d}x) = \Gamma + \mathrm{i}Q, \tag{2.102}$$

where Γ is the *circulation* around C, while Q is the *flux* through C. (These are equal to the jumps in the values of ϕ and ψ respectively as the closed circuit C is traced.) The flux must be zero, as there are not supposed to be any sources or sinks in the flow, and we deduce from (2.101) and (2.102) that

$$b_1 = -\frac{\mathrm{i}\Gamma}{2\pi}. \tag{2.103}$$

Hence equation (2.100) becomes

$$D - \mathrm{i}L = \mathrm{i}\rho U \Gamma, \tag{2.104}$$

and the result (2.94) follows. ∎

2.5 Background material

2.5.1 Integral theorems

Green's Theorem Let the functions $P(x, y)$ and $Q(x, y)$ be continuously differentiable in the region D bounded by the simple closed curve ∂D (traversed in the anti-clockwise sense) with outward unit normal $\boldsymbol{n}$. Then the Divergence Theorem restricted to two dimensions tells us that

$$\iint_D \nabla \cdot (P, Q)\, \mathrm{d}x\mathrm{d}y \equiv \oint_{\partial D} (P, Q) \cdot \boldsymbol{n}\, \mathrm{d}s, \tag{2.105}$$

which may be rewritten as

$$\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \mathrm{d}x\mathrm{d}y \equiv \oint_{\partial D} (P\, \mathrm{d}y - Q\, \mathrm{d}x). \tag{2.106}$$

Fundamental Theorem of Calculus for line integrals For any continuously differentiable scalar function $\phi(x, y)$, we have

$$\int_C \nabla \phi \cdot \mathrm{d}\boldsymbol{x} = \int_C \left(\frac{\partial \phi}{\partial x} \mathrm{d}x + \frac{\partial \phi}{\partial y} \mathrm{d}y \right) = \phi(\boldsymbol{b}) - \phi(\boldsymbol{a}) \tag{2.107}$$

where C is a smooth path joining the two points with position vectors $\boldsymbol{x} = \boldsymbol{a}$ and $\boldsymbol{x} = \boldsymbol{b}$.

2.5.2 Plane polar coordinates

Definition (r, θ) are related to (x, y) by

$$x \equiv r \cos \theta, \qquad y \equiv r \sin \theta. \tag{2.108}$$

The corresponding unit basis vectors are

$$\boldsymbol{e}_r \equiv \boldsymbol{i} \cos \theta + \boldsymbol{j} \sin \theta, \qquad \boldsymbol{e}_\theta \equiv -\boldsymbol{i} \sin \theta + \boldsymbol{j} \cos \theta, \tag{2.109}$$

in terms of the usual Cartesian basis vectors $\{\boldsymbol{i}, \boldsymbol{j}\}$.

Grad, div and curl Given a two-dimensional scalar field $\phi(r, \theta)$ and a vector field $\boldsymbol{u}(r, \theta) = u_r(r, \theta)\boldsymbol{e}_r + u_\theta(r, \theta)\boldsymbol{e}_\theta$, we have

$$\nabla \phi \equiv \frac{\partial \phi}{\partial r} \boldsymbol{e}_r + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \boldsymbol{e}_\theta, \tag{2.110}$$

$$\nabla \cdot \boldsymbol{u} \equiv \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}, \tag{2.111}$$

$$\nabla^2 \phi \equiv \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}, \tag{2.112}$$

$$\nabla \times \boldsymbol{u} \equiv \left(\frac{1}{r} \frac{\partial}{\partial r} (r u_\theta) - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \boldsymbol{k}, \tag{2.113}$$

$$\nabla \times (\phi \boldsymbol{k}) \equiv \frac{1}{r} \frac{\partial \phi}{\partial \theta} \boldsymbol{e}_r - \frac{\partial \phi}{\partial r} \boldsymbol{e}_\theta, \tag{2.114}$$

where $\boldsymbol{k} \equiv \boldsymbol{i} \times \boldsymbol{j} \equiv \boldsymbol{e}_r \times \boldsymbol{e}_\theta$ is a unit vector perpendicular to the (x, y)-plane.

2.5.3 Complex analysis

Standard notation Given a complex number $z = x + \mathrm{i}y$, we define:

- $\operatorname{Re} z = x$, the real part of z;
- $\operatorname{Im} z = y$, the imaginary part of z;
- $\overline{z} = x - \mathrm{i}y$, the complex conjugate of z;
- $|z| = \sqrt{x^2 + y^2}$, the modulus of z.

Note that the complex conjugate commutes with all standard mathematical operations, that is

$$\overline{z_1 + z_2} \equiv \overline{z_1} + \overline{z_2}, \qquad \overline{z_1 z_2} \equiv \overline{z_1}\,\overline{z_2}, \qquad \overline{\mathrm{e}^{z}} \equiv \mathrm{e}^{\overline{z}}, \tag{2.115}$$

and so forth.

Derivatives and holomorphic functions A function $f(z)$ is *holomorphic* (or *analytic*) if it has a well-defined derivative

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{2.116}$$

which is independent of the direction in which the complex variable h approaches zero.

Any holomorphic function has

- *derivatives of all orders*, and
- a *power series expansion*, with positive radius of convergence.

A complex-valued function may be decomposed into its real and imaginary parts, i.e. $f(z) \equiv \phi(x, y) + \mathrm{i}\psi(x, y)$. Then $f(z)$ is holomorphic if and only if the first partial derivatives of ϕ and ψ are continuous and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{2.117}$$

Laurent expansions If a function $f(z)$ is holomorphic on an *annulus*, say $\delta < |z - a| < R$, then it has a convergent *Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z - a)^n. \tag{2.118}$$

The singularity at $z = a$ is

1. *removable* if $b_n = 0$ for all $n < 0$;
2. a *pole of order N* if $b_{-N} \neq 0$ but $b_n = 0$ for all $n < -N$;
3. an *essential singularity* if there does not exist an N such that $b_n = 0$ for all $n < -N$.

In case 2, the **residue** $\operatorname{Res}(f(z); a)$ of the pole is the coefficient b_{-1} of $(z - a)^{-1}$ in the Laurent expansion about $z = a$.

Multifunctions Functions such as $\log z$ and z^α for non-integer α cannot be defined both uniquely and continuously when z is complex. For example,

$$\log z = \log r + \mathrm{i}\theta, \tag{2.119}$$

where (r, θ) are the usual plane polar coordinates, also known as the *modulus* and *argument* of z. To define $\log z$ uniquely, we have to *choose a branch* of θ. For example, if we choose θ to lie in the range $\theta \in [0, 2\pi)$, then $\log z$ is uniquely defined, but discontinuous across the *branch cut* at $\theta = 0$, i.e. along the postive real axis.

By writing

$$z^\alpha \equiv r^\alpha \mathrm{e}^{\mathrm{i}\alpha\theta}, \tag{2.120}$$

we can also define z^α uniquely once we have chosen a branch for θ.

For more complicated multifunctions, see Priestley Chapter 9.

Contour integrals A closed curve C in the complex plane may be parametrised as $z = \gamma(\tau)$, where the real parameter τ lies in some interval $\tau \in [0, c]$ and $\gamma(0) = \gamma(c)$. Then we define the contour integral of $f(z)$ around C as

$$\oint_C f(z)\,\mathrm{d}z := \int_0^c f\big(\gamma(\tau)\big)\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\,\mathrm{d}\tau. \tag{2.121}$$

Cauchy's Theorem If $f(z)$ is holomorphic inside and on the simple close curve C, then

$$\oint_C f(z)\,\mathrm{d}z \equiv 0. \tag{2.122}$$

Deformation Theorem Suppose that the contour $\tilde{C}$ lies inside the contour C, and that the function $f(z)$ is holomorphic on $\tilde{C}$, on C and on the region contained between them. Then the contour C may be deformed onto $\tilde{C}$ without changing the contour integral of $f(z)$:

$$\oint_C f(z)\,\mathrm{d}z \equiv \oint_{\tilde{C}} f(z)\,\mathrm{d}z. \tag{2.123}$$

Cauchy's Residue Theorem Suppose $f(z)$ is holomorphic on and inside a contour C, except for a finite number of poles at the points $a_1, \ldots, a_n$ inside C. Then the contour integral of $f(z)$ around C is equal to $2\pi\mathrm{i}$ times the sum of the residues of all the poles inside C, *i.e.*

$$\oint_C f(z)\,\mathrm{d}z \equiv 2\pi\mathrm{i}\sum_{k=1}^{n} \mathrm{Res}\,(f(z); a_k)\,. \tag{2.124}$$