LECTURE 8: CONFORMAL MAPPING

3.1 Wedges and channels

3.1.1 The basic idea

Suppose we wish to find the flow due to some given singularities (sources, vortices, *etc.*) in a region $R \subset \mathbb{C}$ with impermeable boundary ∂R. The idea is to perform a *conformal mapping* $\zeta = g(z)$ so that the region R in the z-plane is mapped to a much simpler region $\hat{R}$ in the ζ-plane (for example a half-space). As shown schematically in Figure 3.1, any sources, vortices, *etc.* present in R will be mapped to corresponding singularities in $\hat{R}$. Now we can hopefully solve the transformed problem, for example using the method of images, to get the complex potential $W(\zeta)$. Then we just have to invert the conformal mapping to recover the complex potential in the original z-plane, namely

$$w(z) = W\big(g(z)\big). \tag{3.1}$$

We begin by illustrating the general approach using a simple example.

Example 3.1 ***Source in a wedge***
Suppose fluid occupies the wedge-shaped region $0 < \theta < \alpha$, where θ is the usual polar angle, with impermeable walls on $\theta = 0$ and $\theta = \alpha$. We wish to find the flow due to a source of strength Q placed at a point $z = c$ inside the fluid (with $0 < \arg z < \alpha$ and $|c| > 0$).

We recall that the transformation

$$\zeta = g(z) = z^{\pi/\alpha} \tag{3.2}$$

maps the given wedge $0 < \arg z < \alpha$ onto the half-space $\operatorname{Im}\zeta > 0$, and is conformal everywhere inside the wedge except at the origin. The source is mapped to the point $\zeta = c^{\pi/\alpha}$ which must lie somewhere in the half-space $\operatorname{Im}\zeta > 0$.

The transformed problem of a source in a half-space is easily solved by the method of images, and the solution is found to be

$$W(\zeta) = \frac{Q}{2\pi}\log\left(\zeta - c^{\pi/\alpha}\right) + \frac{Q}{2\pi}\log\left(\zeta - \overline{c^{\pi/\alpha}}\right). \tag{3.3}$$

Now reversing the transformation, we find the complex potential

$$w(z) = W\left(z^{\pi/\alpha}\right) = \frac{Q}{2\pi}\log\left(z^{\pi/\alpha} - c^{\pi/\alpha}\right) + \frac{Q}{2\pi}\log\left(z^{\pi/\alpha} - \overline{c^{\pi/\alpha}}\right). \tag{3.4}$$

In particular, if we set $\alpha = \pi/2$, then (3.4) becomes

$$w(z) = \frac{Q}{2\pi}\log\left(z^2 - c^2\right) + \frac{Q}{2\pi}\log\left(z^2 - \bar{c}^2\right). \tag{3.5}$$

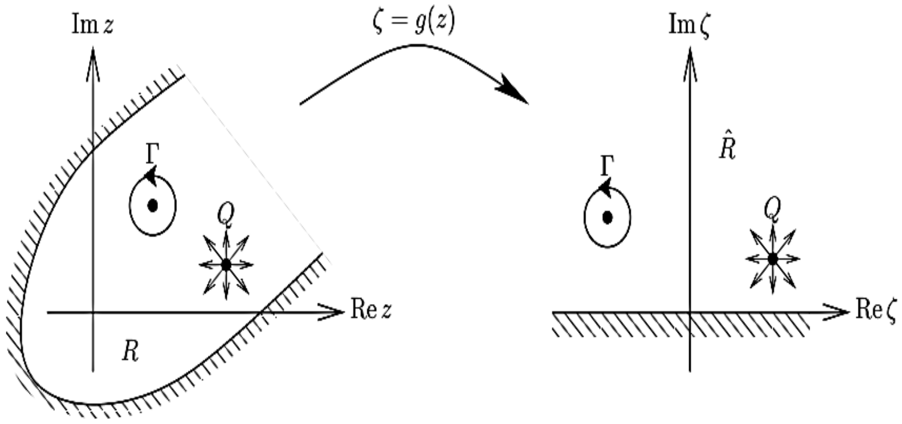

Figure 3.1: Schematic of a conformal mapping $\zeta = g(z)$ mapping a region R in the z-plane to the upper half-plane $\hat{R} = \{\zeta : \text{Im}\,\zeta > 0\}$ in the ζ-plane.

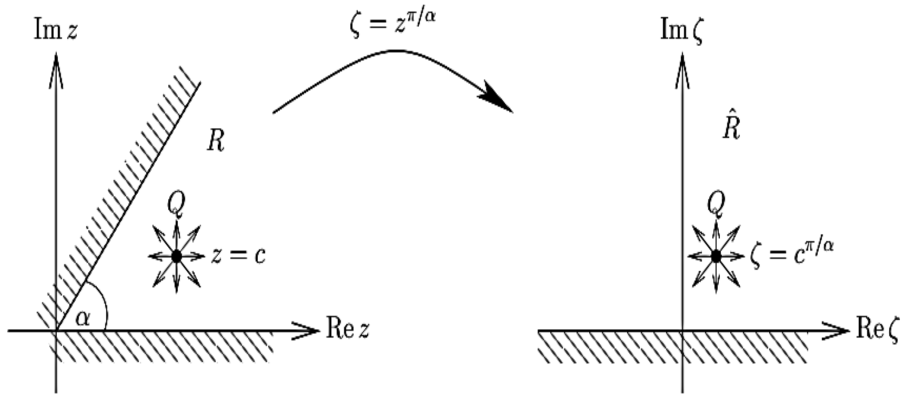

Figure 3.2: Schematic of the conformal mapping $\zeta = z^{\pi/\alpha}$ mapping the wedge $R = \{z : 0 < \arg z < \alpha\}$ in the z-plane to the upper half-plane $\hat{R} = \{\zeta : \text{Im}\,\zeta > 0\}$ in the ζ-plane.

The arguments of the logs *are easily factorised to give*

$$w(z) = \frac{Q}{2\pi}\log(z-c) + \frac{Q}{2\pi}\log(z+c) + \frac{Q}{2\pi}\log(z-\bar{c}) + \frac{Q}{2\pi}\log(z+\bar{c}), \qquad (3.6)$$

which reproduces the result obtained for a quadrant in Section 2 *by generalising the method of images.*

3.1.2 Background theory

The above example illustrates the following important points that underpin the method.

1. **Conformal maps** are defined to be functions $\zeta = g(z)$ where $g(z)$ is *holomorphic* in R and $\mathrm{d}g/\mathrm{d}z \neq 0$ in R. We will be concerned with pairs of domains R and $\hat{R}$ for which the map g defines a bijection between R and $\hat{R}$. The two domains are then said to be *conformally equivalent.*

2. Conformal maps **preserve angles**. We therefore expect the mapping not to be conformal at isolated *corners* in the boundary of R, where the angle is altered by the transformation. In Example 3.1, the mapping $\zeta = g(z)$ is not conformal at $z = 0$, where the angle is transformed from α to π.

3. If the boundary $\partial\hat{R}$ is a streamline in the ζ-plane, then the corresponding boundary ∂R is a streamline in the z-plane (and *vice versa*).

Proof If $\partial\hat{R}$ is a streamline, then $\operatorname{Im} W(\zeta) = C$ for all $\zeta \in \hat{R}$, for some constant C. Since $\partial\hat{R}$ is the image of ∂R under the conformal mapping g, we can write $\partial\hat{R} = g(\partial R) = \{g(z) : z \in \partial R\}$. It follows that $\operatorname{Im} w(z) = \operatorname{Im} W\big(g(z)\big) = C$ for all $z \in \partial R$, and hence that ∂R is a streamline.

The converse result also holds since g is a bijection, so we can use exactly the same argument with g^{-1}. ▮

4. A source of strength Q at $\zeta = g(c) \in \hat{R}$ in the ζ-plane corresponds to a source *of the same strength* Q at $z = c \in R$ in the z-plane (and *vice versa*).

Proof Suppose there is a source of strength Q at $\zeta = g(c) \in \hat{R}$, so that the complex potential takes the form

$$W(\zeta) \sim \frac{Q}{2\pi}\log\big(\zeta - g(c)\big) + \mathrm{O}(1) \quad \text{as } \zeta \to g(c). \tag{3.7}$$

Since $z \mapsto g(z)$ is a bijection from R to $\hat{R}$, $\zeta \to g(c)$ if and only if $z \to c$, and it follows that

$$w(z) = W\big(g(z)\big) \sim \frac{Q}{2\pi}\log\big(g(z) - g(c)\big) + \mathrm{O}(1) \quad \text{as } z \to c. \tag{3.8}$$

Furthermore, Taylor's Theorem gives

$$g(z) - g(c) \sim g'(c)(z-c) + \mathrm{O}\left((z-c)^2\right) \quad \text{as } z \to c, \tag{3.9}$$

and hence

$$w(z) \sim \frac{Q}{2\pi}\log\Big(g'(c)(z-c) + \mathrm{O}\left((z-c)^2\right)\Big) + \mathrm{O}(1) \quad \text{as } z \to c, \tag{3.10}$$

where $g'(c) \neq 0$ since the mapping is supposed to be conformal. Therefore we can expand out the log to obtain

$$w(z) \sim \frac{Q}{2\pi}\log(z-c) + \mathrm{O}(1) \quad \text{as } z \to c, \tag{3.11}$$

which implies the presence of a source of strength Q at $z = c$.

Again, the same argument works in reverse using g^{-1} instead of g. ▮

Obviously, the same conclusion holds for vortices. However, the argument fails if we put a source or a vortex at a point where the mapping is not conformal, for example at an isolated corner in the boundary. Such cases require special treatment and will be avoided in this course.

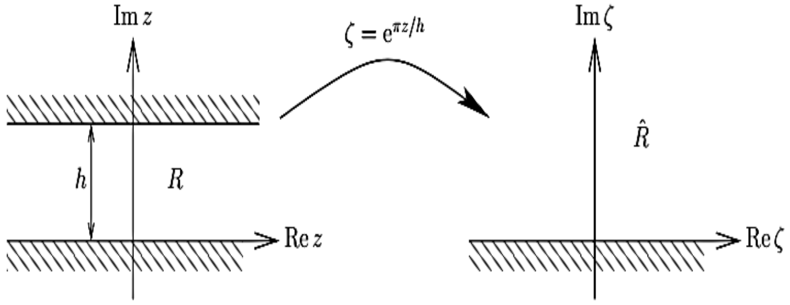

Figure 3.3: Schematic of the exponential conformal mapping $\zeta = e^{\pi z/h}$ mapping the channel $R = \{z : 0 < \operatorname{Im} z < h\}$ in the z-plane to the upper half-plane $\hat{R} = \{\zeta : \operatorname{Im}\zeta > 0\}$ in the ζ-plane.

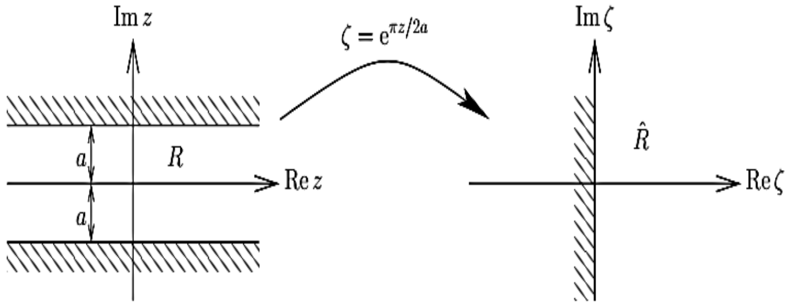

Figure 3.4: Schematic of the exponential conformal mapping $\zeta = e^{\pi z/2a}$ mapping the channel $R = \{z : -a < \operatorname{Im} z < a\}$ in the z-plane to the right half-plane $\hat{R} = \{\zeta : \operatorname{Re}\zeta > 0\}$ in the ζ-plane.

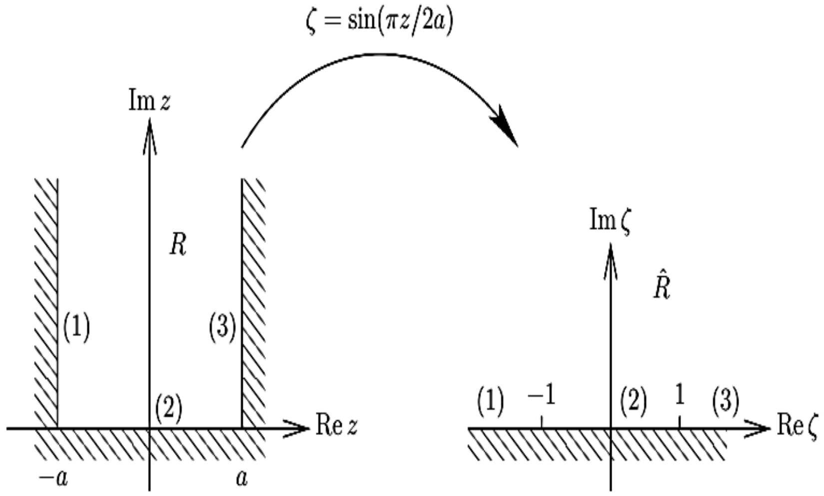

Figure 3.5: Schematic of the semi-infinite channel $R = \{z : \operatorname{Im} z > 0, -a < \operatorname{Re} z < a\}$ being transformed to the half-plane $\hat{R} = \{\zeta : \operatorname{Im} \zeta > 0\}$ by the conformal mapping $\zeta = \sin(\pi z/2a)$.

3.1.3 Other examples of conformal maps

Exponential maps These are used to map a channel to a half-space. To start with, consider a channel of width h occupying the region $0 < \operatorname{Im} z < h$ in the z-plane. The mapping

$$\zeta = g(z) = e^{\pi z/h} \tag{3.12}$$

maps this to the upper half-plane $\operatorname{Im} \zeta > 0$, as shown schematically in Figure 3.3. We can easily see this by writing a typical point in R as $z = x + iy$, where $0 < y < h$. The corresponding point in $\hat{R}$ is $\zeta = e^{\pi x/h} e^{i\pi y/h} = re^{i\theta}$, where $r > 0$ and $0 < \theta < \pi$. Hence $\hat{R}$ is the region $0 < \arg \zeta < \pi$, that is, the upper half-plane $\operatorname{Im} \zeta > 0$.

The exponential mapping is easily generalised to deal with other channel-shaped regions. For example, the shifted channel $-a < \operatorname{Im} z < a$ is transformed to the right half-plane $\operatorname{Re} \zeta > 0$ by the mapping $\zeta = e^{\pi z/2a}$, as shown schematically in Figure 3.4.

Trigonometric maps These are used to map a semi-infinite channel onto a half-space. Consider for example the region R shown in Figure 3.5, where $\operatorname{Im} z > 0$ and $-a < \operatorname{Re} z < a$. The key observation is that, if we define

$$\zeta = g(z) = \sin\left(\frac{\pi z}{2a}\right), \tag{3.13}$$

then the mapping $z \mapsto g(z)$ is conformal everywhere in R and on ∂R, except at the corners $z = \pm a$, where dg/dz is zero. It is plausible, then, that this mapping may have the desired effect of "opening out" the angles of these corners and thus transforming ∂R into a straight line.[1] We can confirm that this is the case by considering the fates of the

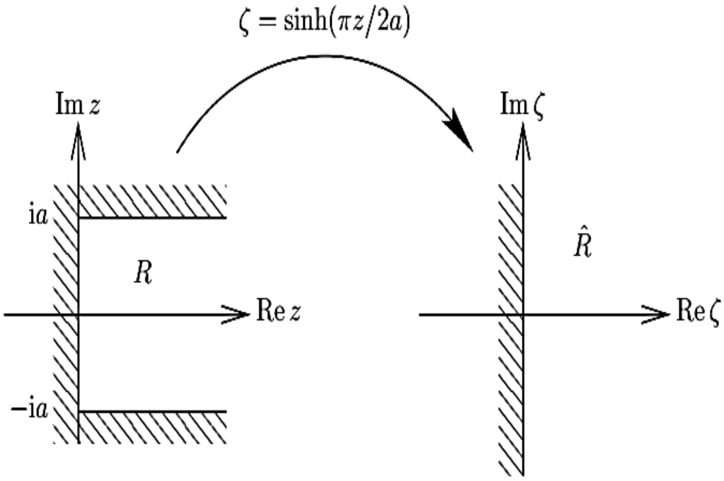

Figure 3.6: Schematic of the semi-infinite channel $R = \{z : \mathrm{Re}\, z > 0, -a < \mathrm{Im}\, z < a\}$ being transformed to the half-plane $\hat{R} = \{\zeta : \mathrm{Re}\,\zeta > 0\}$ by the conformal mapping $\zeta = \sinh(\pi z/2a)$.

three line segments marked (1), (2) and (3) in Figure 3.5.

We can parametrise line (1) by writing $z = -a + \mathrm{i}y$, where $y > 0$. Its image is therefore parametrised by

$$\zeta = \sin\left(-\frac{\pi}{2} + \mathrm{i}\frac{\pi y}{2a}\right) = -\cosh\left(\frac{\pi y}{2a}\right), \tag{3.14}$$

which traces out the segment of the real-ζ-axis from $-\infty$ to -1, as indicated in Figure 3.5. On line (2), we have $z = x \in (-a, a)$ and hence

$$\zeta = \sin\left(\frac{\pi x}{2a}\right), \tag{3.15}$$

which parametrises in the interval $(-1, 1)$ on the real-ζ-axis. Finally, the image of line (3) is given by

$$\zeta = \sin\left(\frac{\pi}{2} + \mathrm{i}\frac{\pi y}{2a}\right) = \cosh\left(\frac{\pi y}{2a}\right), \tag{3.16}$$

where $y > 0$, and this describes the segment $(1, \infty)$ on the real-ζ-axis.

These calculations establish that the boundary ∂R gets mapped to the real axis in the ζ-plane. It only remains to decide whether R is mapped to the upper or the lower half-plane, and this can be determined by considering just one point. For example, the point $z = \mathrm{i} \in R$ is mapped to $\zeta = \mathrm{i}\sinh(\pi/2)$, which has positive imaginary part, and it follows that the image of R is the upper half-plane $\mathrm{Im}\,\zeta > 0$, as indicated in Figure 3.5.

Other semi-infinite channel geometries can be handled using generalisations of (3.13). For example, the channel $\{z : \mathrm{Re}\, z > 0, -a < \mathrm{Im}\, z < a\}$ is transformed to the right half-plane by the mapping

$$\zeta = g(z) = \sinh(\pi z/2a), \tag{3.17}$$

as shown schematically in Figure 3.6. Again, the key to choosing the right map is to identify the corners where $\mathrm{d}g/\mathrm{d}z$ should be zero.

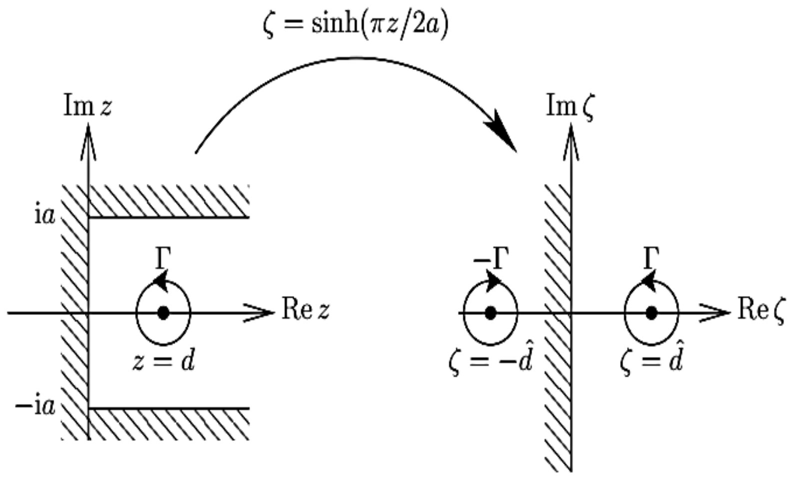

Figure 3.7: Schematic of a vortex at $z = d \in \mathbb{R}^+$ in the semi-infinite channel $R = \{z : \mathrm{Re}\, z > 0, -a < \mathrm{Im}\, z < a\}$ being transformed by the conformal mapping $\zeta = \sinh(\pi z/2a)$.

Example 3.2 ***A vortex in a semi-infinite channel***
Consider the channel depicted in the left-hand z-plane in Figure 3.6. *Now we wish to find the flow caused by the insertion of a vortex of strength* Γ *at the point* $z = d \in \mathbb{R}^+$, *as shown in* Figure 3.7.

We use the mapping (3.17) *to transform the channel on the right half-plane* $\mathrm{Re}\,\zeta > 0$. *The vortex at* $z = d$ *is mapped to* $\zeta = \hat{d} = \sinh(\pi d/2a)$. *We can then satisfy the boundary condition on the imaginary* ζ*-axis by inserting an equal and opposite vortex at the image point* $\zeta = -\hat{d}$, *as shown in* Figure 3.7. *The complex potential is thus given by*

$$W(\zeta) = -\frac{\mathrm{i}\Gamma}{2\pi}\log\left(\zeta - \hat{d}\right) + \frac{\mathrm{i}\Gamma}{2\pi}\log\left(\zeta + \hat{d}\right), \tag{3.18}$$

or, in terms of the original complex variable z,

$$w(z) = W\big(g(z)\big) = \frac{\mathrm{i}\Gamma}{2\pi}\left\{-\log\left(\sinh\left(\frac{\pi z}{2a}\right) - \hat{d}\right) + \log\left(\sinh\left(\frac{\pi z}{2a}\right) + \hat{d}\right)\right\}. \tag{3.19}$$

The velocity components are thus given by

$$u - \mathrm{i}v = \frac{\mathrm{d}w}{\mathrm{d}z} = -\left(\frac{\mathrm{i}\Gamma}{2a}\right)\left(\frac{\cosh(\pi z/2a)\sinh(\pi z/2a)}{\sinh^2(\pi z/2a) - \sinh^2(\pi d/2a)}\right), \tag{3.20}$$

and a little simplification leads to

$$u - \mathrm{i}v = \frac{\mathrm{i}\Gamma}{4a}\left\{\mathrm{cosech}\left(\frac{\pi(z+d)}{2a}\right) - \mathrm{cosech}\left(\frac{\pi(z-d)}{2a}\right)\right\}. \tag{3.21}$$

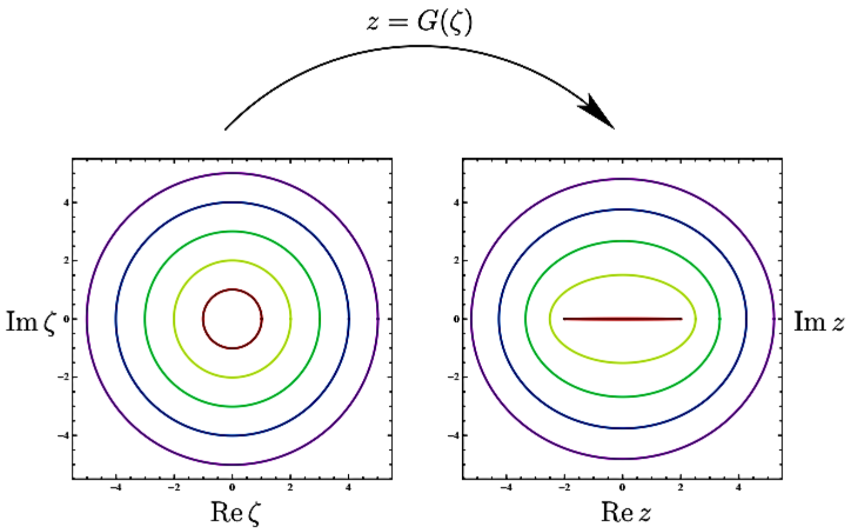

Figure 3.8: The images of circles $|\zeta| = r$ with $r \geqslant a$ under the Joukowski transformation $z = G(\zeta)$ given by (3.22). (Here $a = 1$.)