**Q1. Third order, direct space time method.** i) Derive a third order accurate (time and space) finite difference approximation to the linear advection problem

$$\partial_t\theta + c\partial_x\theta = 0 \tag{1}$$

where $c > 0$ a positive constant flow. The resulting scheme should take the form

$$\frac{1}{\Delta t}(\theta_i^{n+1} - \theta_i^n) = -\frac{c}{\Delta x}(\delta\theta_{i-2}^n + \gamma\theta_{i-1}^n + \beta\theta_i^n + \alpha\theta_{i+1}^n) \tag{2}$$

where $\alpha$, $\beta$, $\gamma$ and $\delta$ are factors that you will determine. Assume a regular grid with index $i$ such that $x_i = i\Delta x$ and $\theta_i = \theta(x_i)$. Hint: You will need higher time derivatives of the above governing equation to eliminate the first and second order time truncation terms.
ii) Derive the discrete flux $F$ that when used in the difference equation

$$\frac{1}{\Delta t}(\theta_i^{n+1} - \theta_i^n) = -\frac{1}{\Delta x}(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}) \tag{3}$$

makes it equivalent to the difference equation (1). Hint: $F$ takes the form

$$F_{i+\frac{1}{2}} = c[\theta_i + d_1(\theta_i - \theta_{i-1}) + d_0(\theta_{i+1} - \theta_i)] \tag{4}$$

where $d_0$ and $d_1$ are functions of the Courant number, $C = \frac{c\Delta t}{\Delta x}$.
iii) Consider this flux in the limit of vanishing Courant number. What discretization does this correspond to (see your previous problem set)?

**Q2. Finite volume method** Again, consider the linear advection problem cast in flux form (3) where $F = c\theta$ with $c > 0$ on a regular grid. We will consider the flux of properties across the point $x = x_{i+\frac{1}{2}}$ as the average of the upstream time-average of

$$F_{i+\frac{1}{2}} = \frac{1}{\Delta t}\int_{x_{i+\frac{1}{2}}-c\Delta t}^{x_{i+\frac{1}{2}}} \theta(x)\,dx \tag{5}$$

i) Consider the distribution of $\theta$ at time $t = n\Delta t$ assuming that $\theta$ is piecewise constant in the finite volume $\Delta x$ around each point $x_i$ (i.e. $\theta$ is constant with value $\theta_i$ between $x_i - \frac{1}{2}\Delta x$ and $x_i + \frac{1}{2}\Delta x$.).
a) Evaluate $F_{i+\frac{1}{2}}$ in equation (5). You may assume that $\Delta t \leq \Delta x/c$.
b) What is this scheme usually called?
c) To make this calculation, why is it useful to assume $\Delta t \leq \Delta x/c$?

d) Now re-evaluate $F_{i+\frac{1}{2}}$ in equation (5), this time assuming $\Delta x/c \leq \Delta t \leq 2\Delta x/c$.
e) Generalize you answers for (a) and (d) so that you can evaluate $F_{i+\frac{1}{2}}$ using one expression assuming $\Delta t \leq 2\Delta x/c$. Hint: you will need to use the min and max functions:

$$
\begin{aligned}
min(a,b) &= \begin{cases} a & \text{if} \quad a \leq b \\ b & \text{if} \quad a > b \end{cases} \\
max(a,b) &= \begin{cases} a & \text{if} \quad a \geq b \\ b & \text{if} \quad a < b \end{cases}
\end{aligned}
$$

ii) Consider the distribution of $\theta$ at time $t = n\Delta t$ to be piecewise linear between the nodes $x_i$.
a) Write down $\theta$ as a function of $x$ in the interval $x_i \leq x \leq x_{i+1}$. Hint: this is simply linear interpolation between the values $\theta_i$ and $\theta_{i+1}$.
b) Evaluate $F_{i+\frac{1}{2}}$ in equation (5) assuming a piecewise linear distribution. You may assume that $\Delta t \leq \frac{1}{2}\Delta x/c$.
c) What is this scheme usually called?
iii) Consider the distribution of $\theta$ at time $t = n\Delta t$ to be piecewise quadratic between the nodes $x_i$.
a) Write down $\theta$ as a function of $x$ in the interval $x_i \leq x \leq x_{i+1}$ by fitting a quadratic function to the nodes $\theta_{i-1}$, $\theta_i$ and $\theta_{i+1}$ (i.e $\theta(x_j) = \theta_j$ at $j = i-1, i, i+1$).
b) Evaluate $F_{i+\frac{1}{2}}$ in equation (5) assuming a piecewise quadratic distribution.
c) In the limit of vanishing time-step, what scheme does the flux in (b) approach?
iv) Again, consider the distribution of $\theta$ at time $t = n\Delta t$ to be piecewise quadratic in the interval $x_i \leq x \leq x_{i+1}$ and to take the form:

$$
\theta(x) = \alpha + 2\beta\frac{(x - x_{i+\frac{1}{2}})}{\Delta x} + 3\gamma\frac{(x - x_{i+\frac{1}{2}})^2}{\Delta x^2}. \tag{6}
$$

a) Find $\alpha$, $\beta$ and $\gamma$ so that the spatial average over each finite volume ($\Delta x$) around $x_{i-1}$, $x_i$ and $x_{i+1}$ equals $\theta_{i-1}$, $\theta_i$ and $\theta_{i+1}$ respectively. Note that this is different to fitting the quadratic function at the nodes as you did in part (iii).
b) Evaluate $F_{i+\frac{1}{2}}$ in equation (5) using the "finite volume" representation from (a).
c) What is this scheme usually called?

**Q3. Discrete conservation of variance** The average and difference operators are

$$\begin{aligned}\overline{\theta}^i &= \frac{1}{2}\left(\theta_{i+\frac{1}{2}} + \theta_{i-\frac{1}{2}}\right)\\ \delta_i\theta &= \theta_{i+\frac{1}{2}} - \theta_{i-\frac{1}{2}}\end{aligned}$$

a) Prove the discrete product rule

$$\delta_i(\overline{\theta}^i U) \;=\; \overline{U\delta_i\theta}^i + \theta\delta_i U.$$

b) Prove the discrete product rule

$$\delta_i(\theta\phi) \;=\; \overline{\theta}^i\delta_i\phi + \overline{\phi}^i\delta_i\theta.$$

c) A scalar advection equation and continuity equation are discretized

$$\begin{aligned}\Delta x\Delta y\partial_t\theta + \delta_i(\overline{\theta}^i U\Delta y) + \delta_j(\overline{\theta}^j V\Delta x) &= 0\\ \delta_i(U\Delta y) + \delta_j(V\Delta x) &= 0.\end{aligned}$$

Prove that the global integral of variance ($\int\int \theta^2\, dx\, dy$) is conserved given no normal flow at domain boundaries. Assume perfect treatment of the time derivative.