MATHEMATICAL MODELS OF PHYSICAL SYSTEMS

DIFFERENTIAL EQUATIONS:

A mathematical model is a description of a system using mathematical concepts and language. The process of developing a mathematical model is termed mathematical modeling. Mathematical models are used in the natural sciences (such as physics, biology, earth science, meteorology) and engineering disciplines (such as computer science, artificial intelligence), as well as in the social sciences (such as economics, psychology, sociology, political science). Physicists, engineers, statisticians, operations research analysts, and economists use mathematical models most extensively. A model may help to explain a system and to study the effects of different components, and to make predictions about behaviour.

Mathematical models can take many forms, including dynamical systems, statistical models, differential equations, or game theoretic models. These and other types of models can overlap, with a given model involving a variety of abstract structures. In general, mathematical models may include logical models. In many cases, the quality of a scientific field depends on how well the mathematical models developed on the theoretical side agree with results of repeatable experiments. Lack of agreement between theoretical mathematical models and experimental measurements often leads to important advances as better theories are developed.

Mathematical models are usually composed of relationships and variables. Relationships can be described by operators, such as algebraic operators, functions, differential operators, etc. Variables are abstractions of system parameters of interest, that can be quantified. Several classification critera can be used for mathematical models according to their structure:

Linear vs. nonlinear: If all the operators in a mathematical model exhibit linearity, the resulting mathematical model is defined as linear. A model is considered to be nonlinear otherwise. The definition of linearity and nonlinearity is dependent on context, and linear models may have nonlinear expressions in them. For example, in a statistical linear model, it is assumed that a relationship is linear in the parameters, but it may be nonlinear in the predictor variables. Similarly, a differential equation is said to be linear if it can be written with linear differential operators, but it can still have nonlinear expressions in it. In a mathematical programming model, if the objective functions and constraints are represented entirely by linear equations, then the model is regarded as a linear model. If one or more of the objective functions or constraints are represented with a nonlinear equation, then the model is known as a nonlinear model.

Nonlinearity, even in fairly simple systems, is often associated with phenomena such as chaos and irreversibility. Although there are exceptions, nonlinear systems and models tend to be more difficult to study than linear ones. A common approach to nonlinear problems is linearization, but this can be problematic if one is trying to study aspects such as irreversibility, which are strongly tied to nonlinearity.

Static vs. dynamic: A dynamic model accounts for time-dependent changes in the state of the system, while a static (or steady-state) model calculates the system in equilibrium, and thus is time-invariant. Dynamic models typically are represented by differential equations.

Explicit vs. implicit: If all of the input parameters of the overall model are known, and the output parameters can be calculated by a finite series of computations (known as linear programming, not to be confused with linearity as described above), the model is said to be explicit. But sometimes it is the output parameters which are known, and the corresponding inputs must be solved for by an iterative procedure, such as Newton's method (if the model is linear) or Broyden's method (if non-linear). For example, a jet engine's physical properties such as turbine and nozzle throat areas can be explicitly calculated given a design thermodynamic cycle (air and fuel flow rates, pressures, and temperatures) at a specific flight condition and power setting, but the engine's operating cycles at other flight conditions and power settings cannot be explicitly calculated from the constant physical properties.

Discrete vs. continuous: A discrete model treats objects as discrete, such as the particles in a molecular model or the states in a statistical model; while a continuous model represents the objects in a continuous manner, such as the velocity field of fluid in pipe flows, temperatures and stresses in a solid, and electric field that applies continuously over the entire model due to a point charge.

Deterministic vs. probabilistic (stochastic): A deterministic model is one in which every set of variable states is uniquely determined by parameters in the model and by sets of previous states of these variables; therefore, a deterministic model always performs the same way for a given set of initial conditions. Conversely, in a stochastic model—usually called a "statistical model"—randomness is present, and variable states are not described by unique values, but rather by probability distributions.

Deductive, inductive, or floating: A deductive model is a logical structure based on a theory. An inductive model arises from empirical findings and generalization from them. The floating model rests on neither theory nor observation, but is merely the invocation of expected structure. Application of mathematics in social sciences outside of economics has been criticized for unfounded models.[1] Application of catastrophe theory in science has been characterized as a floating model.

A differential equation is a mathematical equation that relates some function with its derivatives. In applications, the functions usually represent physical quantities, the derivatives represent their rates of change, and the equation defines a relationship between the two. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology.

In pure mathematics, differential equations are studied from several different perspectives, mostly concerned with their solutions—the set of functions that satisfy the equation. Only the simplest differential equations are solvable by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form.

If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a given degree of accuracy.

Differential equations first came into existence with the invention of calculus by Newton and Leibniz. In Chapter 2 of his 1671 work "Methodusfluxionum et SerierumInfinitarum",[1] Isaac Newton listed three kinds of differential equations:

\frac {dy}{dx} = f(x)

$$\frac {dy}{dx} = f(x,y)$$

$$x_1 \frac {\partial y}{\partial x_1} + x_2 \frac {\partial y}{\partial x_2} = y$$

He solves these examples and others using infinite series and discusses the non-uniqueness of solutions.

Jacob Bernoulli proposed the Bernoulli differential equation in 1695.This is an ordinary differential equation of the form

$$y'+ P(x)y = Q(x)y^n\,$$

for which the following year Leibniz obtained solutions by simplifying it

Historically, the problem of a vibrating string such as that of a musical instrument was studied by Jean le Rondd'Alembert, Leonhard Euler, Daniel Bernoulli, and Joseph-Louis LagrangeIn 1746, d'Alembert discovered the one-dimensional wave equation, and within ten years Euler discovered the three-dimensional wave equation

The Euler–Lagrange equation was developed in the 1750s by Euler and Lagrange in connection with their studies of the tautochrone problem. This is the problem of determining a curve on which a weighted particle will fall to a fixed point in a fixed amount of time, independent of the starting point.

Lagrange solved this problem in 1755 and sent the solution to Euler. Both further developed Lagrange's method and applied it to mechanics, which led to the formulation of Lagrangian mechanics.

Fourier published his work on heat flow in Théorieanalytique de la chaleur (The Analytic Theory of Heat), in which he based his reasoning on Newton's law of cooling, namely, that the flow of heat between two adjacent molecules is proportional to the extremely small difference of their temperatures. Contained in this book was Fourier's proposal of his heat equation for conductive diffusion of heat. This partial differential equation is now taught to every student of mathematical physics.

TRANSFER FUNCTION AND BLOCK DIAGRAM OF SIMPLE ELECTRICAL NETWORK:

Transfer function is a form of system representation establishing a viable definition for a function that algebraically relates a system's output to its input. This function allows separation of the input, system, and output into three separate and distinct parts. It also allows to algebraically combines mathematical representations of subsystems to yield a total system representation.

In any problem, the designers must first decide what the input and output should be. In this network, several variables could have been chosen to be the output, for example, the inductor voltage, the capacitor voltage, the resistor voltage, or the current. The problem statement, however, is clear in this case: we are to treat the capacitor voltage as the output and the applied voltage as the input.

Summing the voltages around the loop, assuming zero initial conditions, yields the integrodifferential equation for this network as

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t) \qquad Eq.1$$

Changing the variables form current to charge using i(t) = dq(t)/dt yields

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t) \qquad Eq.2$$

From the voltage-charge relationship for a capacitor in a table shown in Figure 1,

$$q(t) = Cv_C(t) \qquad Eq.3$$

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance Z(s) = V(s)/I(s)	Admittance Y(s) = I(s)/V(s)
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d2q(t)}{dt2}$	Ls	$\frac{1}{Ls}$

Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Substituting Equation 3 into Equation 2 yields

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t) \qquad Eq.4$$

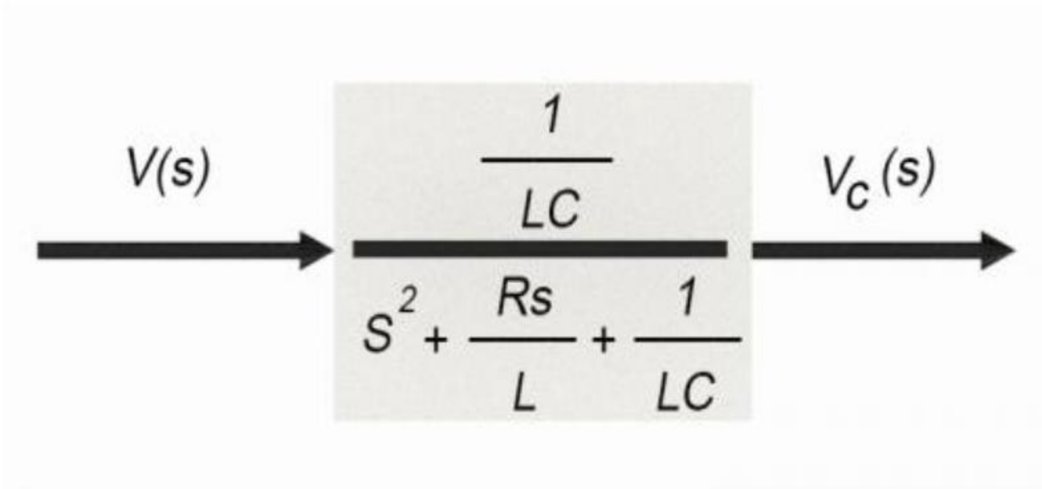
Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying yields

$$(LCs^2 + RCs + 1)V_C(s) = V_s \qquad Eq.5$$

Solving for the transfer function, $V_C(s)/V(s)$, we obtain

$$\frac{V_C(s)}{V(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \qquad Eq.6$$

as shown in.



Transfer Function of the System

TRANSLATIONAL AND ROTATIONAL MECHANICAL SYSTEM:

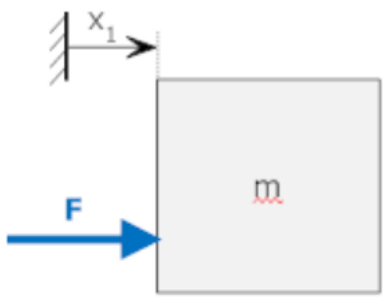
There are three fundamental physical elements that make up translating mechanical system: inertia elements, springs and friction elements. The relationships between force and position (or its derivatives) for these elements are described below.

Contents

- Inertia Elements
- Springs
- Friction Elements
- Key Points

Intertia Elements

In translating mechanical systems, the inertia elements are masses. A mass will be drawn with a coordinate system as in the drawing below.

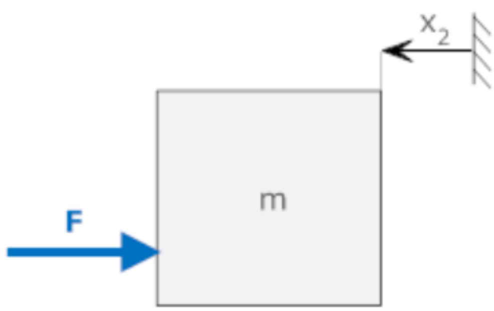


In this drawing the coordinate along which the mass translates is x_1, and x_1 is defined relative to a fixed reference (the vertical line with hash marks to the left). This drawing also shows a force, F, acting on the mass towards the right. Since the force tends to move the mass in the direction of x_1, we can write the well known relationship:

$$F = ma_1 = m\frac{d^2x_1}{dt^2}$$

where a_1 is the second derivative of position with respect to time. The SI units for a mass are kilograms (kg).

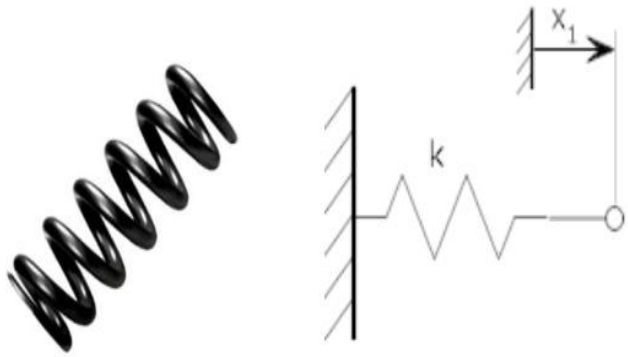
It is important to realize that the defined direction is arbitrary, we could just as easily define the direction in the opposite way. In this case we must change the sign in our equation. The behavior of the system is unchanged regardless of the defined positive direction.



$$F = -ma_2 = -m\frac{d^2x_2}{dt^2}$$

Energy may be stored in a mass as its velocity changes.

A linear translating spring is an element that is deformed (shortened or lengthened) in direct proportion to the amount of force applied. Ideal springs have no mass. A photo of a typical spring is shown below along with a schematic representation.



In this diagram the end of spring k_1 is determined by the position specified by x_1. Unless otherwise specified, the position of x_1 is zero when the spring is relaxes (i.e., no force is applied to it).

If we apply a force to the spring to the right, the spring elongates by an amount Δx such that:

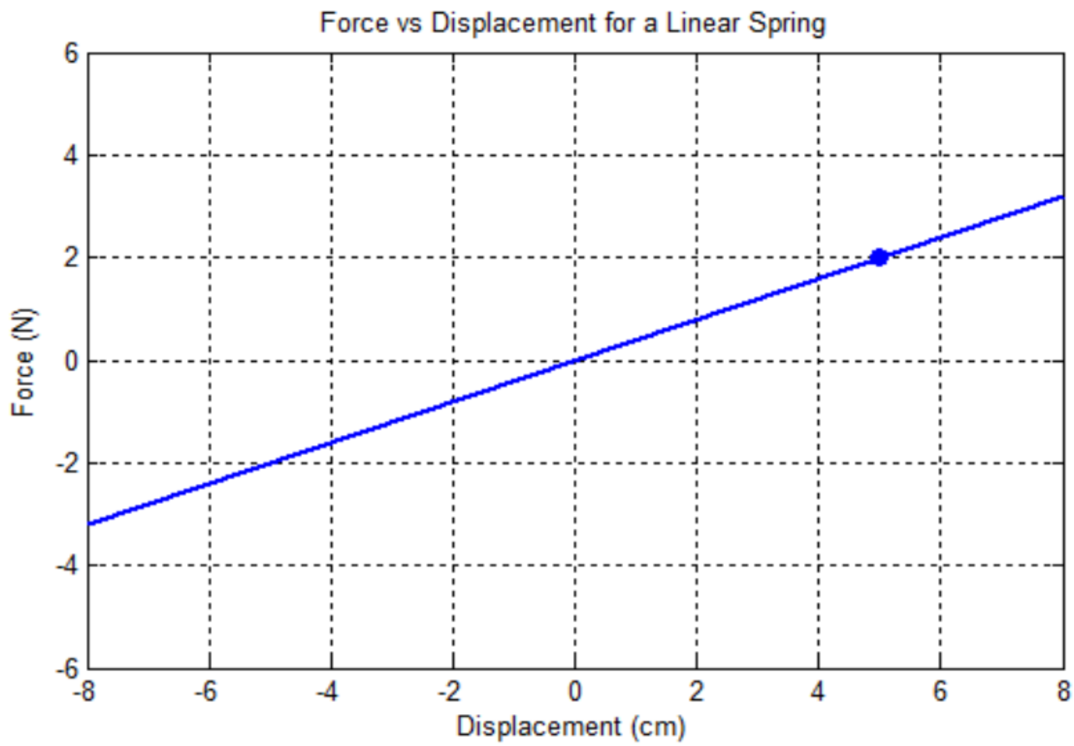
$$F = k\Delta x$$

Since the displacement, x_1, is defined to be zero when the spring is relaxed, we can write:

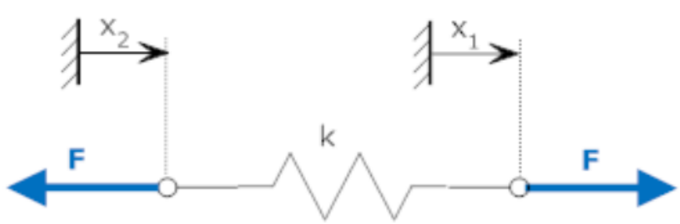
$$F = kx_1$$

The elongation is *proportional* to the force. This is the defining relationship for a *linear* spring. The SI units for the spring consant, k_1, are Newtons/meter (N/m). A stiffer spring has a larger spring constant; it takes more force (Newtons) to obtain a given compression. A graph of force vs. displacement for a linear spring is a straight line with a slope equal to the spring constant. In the graph below, it takes 2 N to stretch the spring 5 cm (as shown by the circle on the graph, so the spring constant is

k = 2 N/0.05 m = 40 N/m.



If neither side of the spring is attached to a fixed reference, the situation is slightly more complex. The forces on both ends of the spring must be equal in magnitude and opposite in direction. This way there is no net force on the spring - if there was a net force there would be infinite acceleartion, because the mass of our ideal spring is zero.

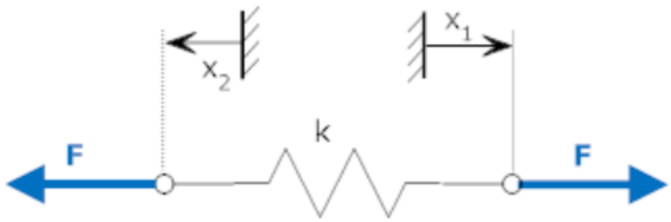


Since the elongation of the spring is now x_1-x_2, we write

$$F = k(x_1 - x_2)$$

In this case the force, F, increases as x_1 increases (because an increase in x_1 causes elongation of the spring), but decreases as x_2 increases (because increasing x_2 shortens the spring). This is an example of superposition; because the system is linear the inputs (x_1 and x_2) can be considered independently.

As with the mass (above) the defined directions are arbitrary. We could have chosen directions as shown below,



which results in an equation where x1 and x2 are added (now an increase in either increases the elongation and, therefore, the force).

$$F = k(x_1 + x_2)$$

Energy may be stored in a spring as its dimension changes. This energy may be released at a later date.

Newton's second law states that an object accelerates in the direction of an applied force, and that this acceleration is inversely proportional to the force, or

$$\sum_{\text{all external}} F = m \cdot a$$

we will bring the right-hand side to the left and express this as

$$\sum_{\text{all external}} F - m \cdot a = 0$$

If we consider the m·a term to be a force, we are left with D'Alembert's law

$$\sum_{\text{all}} F = 0$$

We will call the m·a term D'Alembert's force. It is an inertial force that arises when you try to accelerate a mass. To visualize this consider pushing against a mass (in the absence of friction) with your hand in the positive direction. Your hand experiences a force in the direction opposite to that of the direction of the force (this is the -m·a term). The inertial force is always in a direction opposite to the defined positive direction (see the first example below).

Springs

A rotational spring is an element that is deformed (wound or unwound) in direct proportion to the amount of torque applied. Ideal springs have no inertia. The relationship between torque, spring constant and angle is given by:

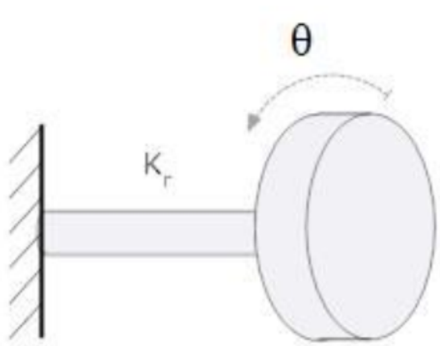
$\tau = K_r\theta$ (*Translating system equivalent:* $f = m\ddot{x} = ma$)

A photo of typical rotational springs is shown. You should be able to visualize how an applied torque to wind (or unwind) the spring would be opposed.

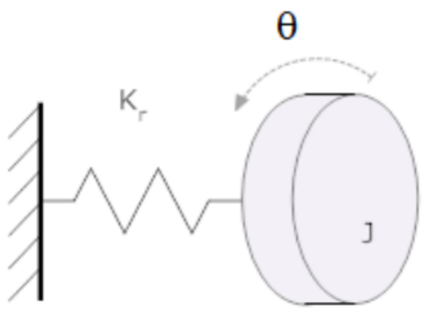
A shaft connected between two elements can also act as a rotational spring. Click the start button to see an animation of a flexible shaft (i.e., rotational spring) that is fixed on the left but free to rotate on the right. Note how the shaft twists as the flywheel (with moment of inertia, J), turns (however the rotation doesn't slow because there is no friction).

In practice a rotational spring is drawn as a shaft (with an associated spring constant) or with the same symbol as a translating spring, but with the spring constant as a capital letter with a subscript. Both images below represent the system from the animation, with theta defined positive in the counterclockwise direction.

Flexible Shaft



Spring



Friction Elements

As with the translating systems, friction is the most difficult of the three elements to model accurately and we will generally only consider viscous friction. The constitutive equation relating angular velocity, torque and friction coefficient is

$\tau = B_r\dot{\theta} = B_r\omega$ (*Translating system equivalent:* $f = b\dot{x} = bv$)

A rotation friction element often consists of an object moving in a fluid, very similar to the translating dashpot, but with a rotary motion. An example of such a device is used in some exercise equipment. The photo below shows an exercise trainer with a paddlewheel in liquid (the blue circular object). As you exercise, the vanes of the paddlewheel move against the fluid, creating a torque in resistance to your motion.

There are several ways of drawing these friction elements. Consider the animation shown below with one flywheel (J_1) attached by a flexible shaft (K_r) to a wall (as before), but now there is a second flywheel (J_2) that is driven by friction between the two flywheels (B_{r1}), and friction to the stationary ground (B_{r2}). J_1 starts with an initial value, so it starts oscillating back and forth as the simulation begins. This drives J_2, through B_{r1}, but the energy in the system decays over time because energy is lost to the friction.

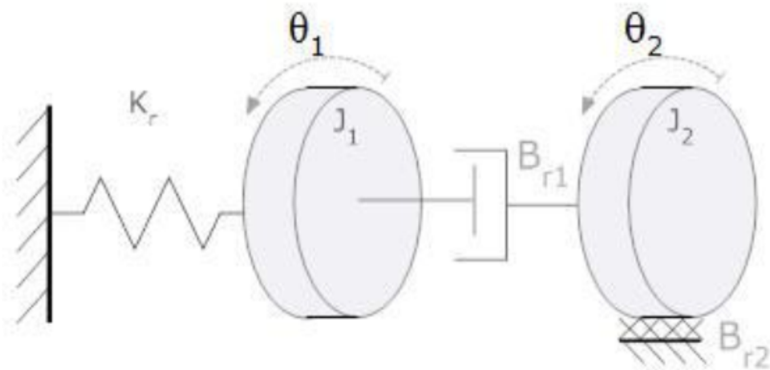
KrJ1Br1Br2J2

This system can be shown schematically in a few ways. In the first diagram below, the shaft is shown schematically as a spring, the friction B_{r1} is drawn as a dashpot, while the friction B_{r2} is shown as hash marks against ground. In the second diagram the shaft is not drawn as a spring but is shown to be a spring with the label K_r; both friction elements are shown as dashpots. It is important to be comfortable with both drawings (and others that could be drawn).

K_r=Spring

B_{r1}=dashpot

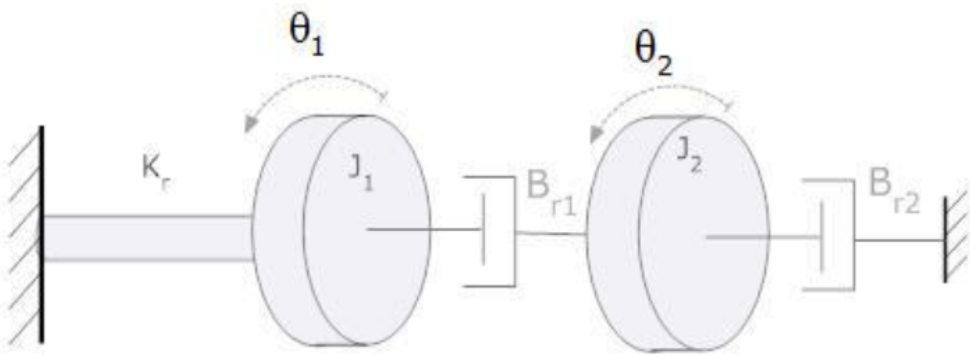
B_{r2}=hashmarks



K_r=Flexible Shaft

B_{r1}=dashpot

B_{r2}=dashpot



Key points

Three elements were introduced, springs, friction elements and inertial elements (masses). An ideal linear spring has no mass and a linear relationship between force and elongation. For viscous friction there is a linear relationship between force and velocity. Friction may either be between two surfaces (depicted as hash marks) or between two objects (depicted as a dashpot). An ideal dashpot is also massless. Masses have a linear relationship between force and acceleration.

Key Concept: Constitutive Equations for Rotating Mechanical Elements

Spring:	$\tau = K_r\theta$
Friction:	$\tau = B_r\dot{\theta} = B_r\omega$
Inertia:	$\tau = J\ddot{\theta} = J\alpha$