CONSTRUCTION OF ROOT LOCI:

To facilitate the application of the root-locus method for systems of higher order than 2nd, rules can be established. These rules are based upon the interpretation of the angle condition and the analysis of the characteristic equation. The rules presented aid in obtaining the root locus by expediting the manual plotting of the locus. But for automatic plotting using a computer these rules provide checkpoints to ensure that the solution is correct.

Though the angle and magnitude conditions can also be applied to systems having dead time, in the following we restrict to the case of the open-loop rational transfer functions according to Eq.

or

$$G_0(s) = k_0 \frac{b_0 + b_1 a + \ldots + b_{m-1}s^{m-1} + s^m}{a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + s^n} = k_0 \frac{N_0(s)}{D_0(s)} .$$

As this transfer function can be written in terms of poles and zeros s_{P_ν} and s_{Z_μ} ($\nu = 1, 2, \ldots n$; $\mu = 1, 2, \ldots, m$) $G_0(s)$ can be represented by their magnitudes and angles

$$G_0(s) = k_0 \frac{|s - s_{Z_1}|\, e^{j\varphi_{Z_1}} |s - s_{Z_2}|\, e^{j\varphi_{Z_2}} \ldots |s - s_{Z_m}|\, e^{j\varphi_{Z_m}}}{|s - s_{P_1}|\, e^{j\varphi_{P_1}} |s - s_{P_2}|\, e^{j\varphi_{P_2}} \ldots |s - s_{P_n}|\, e^{j\varphi_{P_n}}}$$

or

$$G_0(s) = k_0 \frac{\prod_{\mu=1}^{m} |s - s_{Z_\mu}|}{\prod_{\nu=1}^{n} |s - s_{P_\nu}|} e^{j\left(\sum_{\mu=1}^{m} \varphi_{Z_\mu} - \sum_{\nu=1}^{n} \varphi_{P_\nu}\right)} .$$

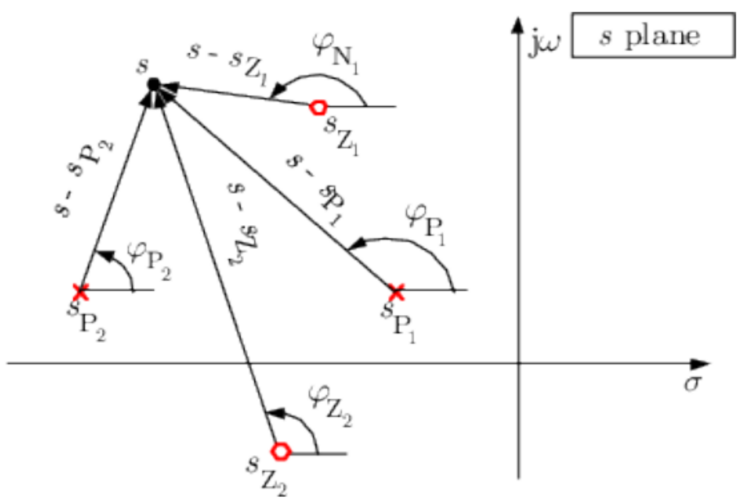
From Eq. (6.8) the *magnitude condition*

$$\frac{\prod_{\mu=1}^{m} |s - s_{Z_\mu}|}{\prod_{\nu=1}^{n} |s - s_{P_\nu}|} = \frac{1}{k_0}$$

and from Eq. the *angle condition*

$$\varphi(s) = \sum_{\mu=1}^{m} \varphi_{Z_\mu} - \sum_{\nu=1}^{n} \varphi_{P_\nu} = \pm 180°(2k+1) \quad \text{for} \quad k = 0, 1, 2, \ldots$$

follows. Here φ_{Z_μ} and φ_{P_ν} denote the angles of the complex values $(s - s_{Z_\mu})$ and $(s - s_{P_\nu})$, respectively. All angles are considered positive, measured in the counterclockwise sense. If for each point the sum of these angles in the s plane is calculated, just those particular points that fulfil the condition in Eq. are points on the root locus. This principle of constructing a root-locus curve - as shown in Figure is mostly used for automatic root-locus plotting.



Pole-zero diagram for construction of the root locus

In the following the most important *rules for the construction of root loci* for $k_0 > 0$ are listed:

Symmetry

As all roots are either real or complex conjugate pairs so that the root locus is symmetrical to the real axis.

Number of branches

The number of branches of the root locus is equal to the number of poles n of the open-loop transfer function.

Locus start and end points

The locus starting points ($k_0 = 0$) are at the open-loop poles and the locus ending points ($k_0 = \infty$) are at the open-loop zeros. $(n - m)$ branches end at infinity. The number of starting branches from a pole and ending branches at a zero is equal to the multiplicity of the poles and zeros, respectively. A point at infinity is considered as an equivalent zero of multiplicity equal to $n - m$.

Real axis locus

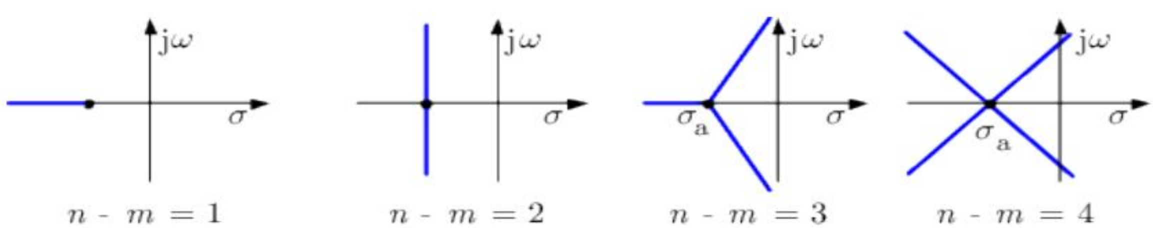
If the total number of poles and zeros to the right of a point on the real axis is odd, this point lies on the locus.

Asymptotes

There are $n - m$ asymptotes of the root locus with a slope of

$$\alpha_k = \arg s = \frac{\pm 180°(2k+1)}{n-m}.$$

For $(n - m) = 1, 2, 3$ and 4 one obtains the asymptote configurations as shown in Figure 6.4.



Asymptote configurations of the root locus

Real axis intercept of the asymptotes

The real axis crossing $(\sigma_a, j0)$ of the asymptotes is at

$$\sigma_a = \frac{1}{n-m}\left\{\sum_{\nu=1}^{n} \operatorname{Re} s_{p_\nu} - \sum_{\mu=1}^{m} \operatorname{Re} s_{z_\mu}\right\}.$$

Breakaway and break-in points on the real axis

At least one breakaway or break-in point $(\sigma_B, j0)$ exists if a branch of the root locus is on the real axis between two poles or zeros, respectively. Conditions to find such real points are based on the fact that they represent multiple real roots. In addition to the characteristic equation for multiple roots the condition

$$\frac{d}{ds}[1 + G_0(s)] = \frac{d}{ds}G_0(s) = 0.$$

must be fulfilled, which is equivalent to

$$\sum_{\nu=1}^{n} \frac{1}{s - s_{p_\nu}} = \sum_{\mu=1}^{m} \frac{1}{s - s_{z_\mu}}$$

for $s = \sigma_B$. If there are no poles or zeros, the corresponding sum is zero.

Complex pole/zero angle of departure/entry

The angle of departure of pairs of poles with multiplicity r_{P_ϱ} is

$$\varphi_{P_\varrho,D} = \frac{1}{r_{P_\varrho}} \left\{ -\sum_{\substack{\nu=1 \\ \nu\neq\varrho}}^{n} \varphi_{P_\nu} + \sum_{\mu=1}^{m} \varphi_{Z_\mu} \pm 180^\circ(2k+1) \right\}$$

and the angle of entry of the pairs of zeros with multiplicity r_{Z_ϱ}

$$\varphi_{Z_\varrho,E} = \frac{1}{r_{Z_\varrho}} \left\{ -\sum_{\substack{\mu=1 \\ \mu\neq\varrho}}^{m} \varphi_{Z_\mu} + \sum_{\nu=1}^{n} \varphi_{P_\nu} \pm 180^\circ(2k+1) \right\} .$$

Rule 9 Root-locus calibration

The labels of the values of k_0 can be determined by using

$$k_0 = \frac{\prod_{\nu=1}^{n} |s - s_{P_\nu}|}{\prod_{\mu=1}^{m} |s - s_{Z_\mu}|} .$$

For $m = 0$ the denominator is equal to one.

Asymptotic stability

The closed loop system is asymptotically stable for all values of k_0 for which the locus lies in the left-half s plane. From the imaginary-axis crossing points the critical values $k_{0_{crit}}$ can be determined.

The rules shown above are for positive values of k_0. According to the angle condition of for negative values of k_0 some rules have to be modified. In the following these rules are numbered as above but labelled by a *.

Locus start and end points

The locus starting points ($k_0 = 0$) are at the open-loop poles and the locus ending points ($k_0 = -\infty$) are at the open-loop zeros. $(n - m)$ branches end at infinity. The number of starting branches from a pole and ending branches at a zero is equal to the multiplicity of the poles and zeros, respectively. A point at infinity is considered as an equivalent zero of multiplicity equal to $n - m$.

Real axis locus

If the total number of poles and zeros to the right of a point on the real axis is even including zero, this point lies on the locus.

Asymptotes

There are $n - m$ asymptotes of the root locus with a slope of

$$\alpha_k = \arg s = \frac{\pm 360^\circ k}{n - m} .$$

Complex pole/zero angle of departure/entry

The angle of departure of pairs of poles with multiplicity r_{P_ϱ} is

$$\varphi_{\mathrm{P}_\varrho,\mathrm{D}} = \frac{1}{r_{\mathrm{P}_\varrho}} \left\{ -\sum_{\substack{\nu=1 \\ \nu \neq \varrho}}^{n} \varphi_{\mathrm{P}_\nu} + \sum_{\mu=1}^{m} \varphi_{\mathrm{Z}_\mu} \pm 360^\circ k \right\}$$

and the angle of entry of the pairs of zeros with multiplicity r_{Z_ϱ}

$$\varphi_{Z_\varrho,E} = \frac{1}{r_{Z_\varrho}} \left\{ -\sum_{\substack{\mu=1 \\ \mu\neq\varrho}}^{m} \varphi_{Z_\mu} + \sum_{\nu=1}^{n} \varphi_{P_\nu} \pm 360^\circ k \right\} .$$

The root-locus method can also be applied for other cases than varying k_0. This is possible as long as $G_0(s)$ can be rewritten such that the angle condition according to Eq. and the rules given above can be applied. This will be demonstrated in the following two examples.

Given the closed-loop characteristic equation

$$a_0 + a_1 s + \ldots + a_{n-1} s^{n-1} + s^n = 0 ,$$

the root locus for varying the parameter a_1 is required. The characteristic equation is therefore rewritten as

$$1 + a_1 \frac{s}{a_0 + a_2 s^2 + \ldots + s^n} = 0 .$$

This form then correspondents to the standard form

$$1 + G_0(s) = 1 + a_1 \frac{N_0(s)}{D_0(s)} = 0$$

to which the rules can be applied. ■

Given the closed-loop characteristic equation

$$s^3 + (3 + \alpha) s^2 + 2s + 4 = 0 ,$$

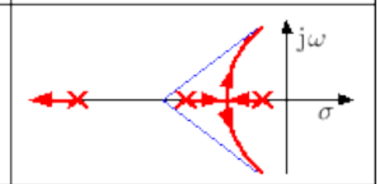
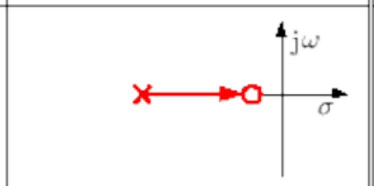
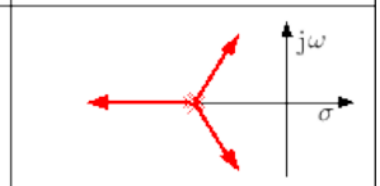
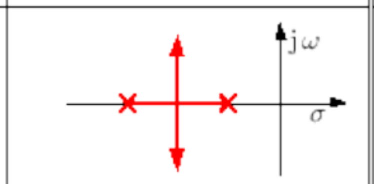
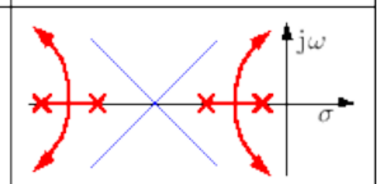
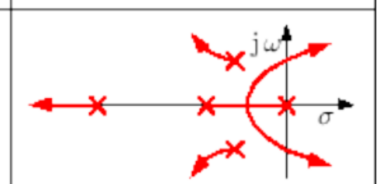
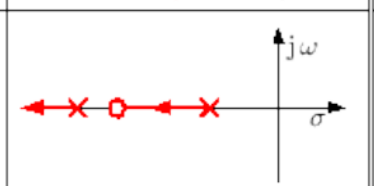
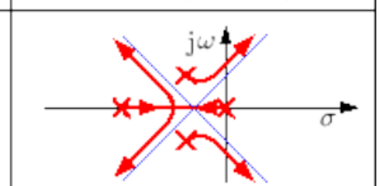
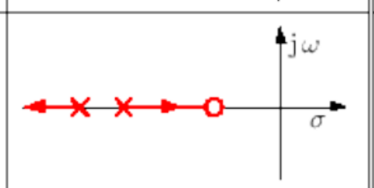
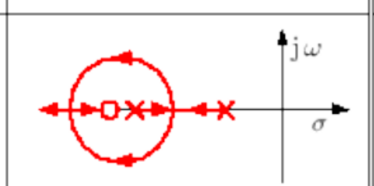
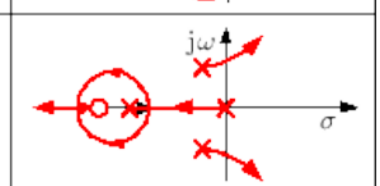
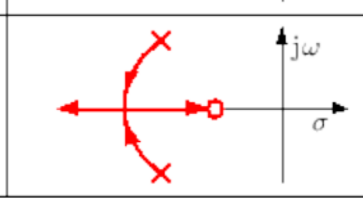
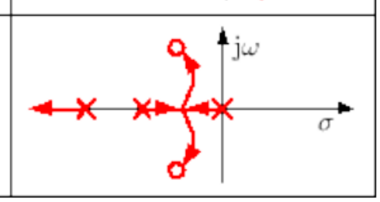
it is required to find the effect of the parameter α on the position of the closed-loop poles. The equation is rewritten into the desired form

$$1 + \alpha \frac{s^2}{s^3 + 3s^2 + 2s + 4} = 0 .$$

■

Using the rules 1 to 10 one can easily predict the geometrical form of the root locus based on the distribution of the open-loop poles and zeros. Table 6.2 shows some typical distributions of open-loop poles and zeros and their root loci.

Typical distributions of open-loop poles and zeros and the root loci

No.	root locus	No.	root locus
1		9	
2		10	
3		11	
4		12	
5		13	
6		14	
7		15	
8		16	

For the qualitative assessment of the root locus one can use a physical analogy. If all open-loop poles are substituted by a negative electrical charge and all zeros by a commensurate positive one and if a massless negative charged particle is put onto a point of the root locus, a movement is observed. The path that the particle takes because of the interplay between the repulsion of the poles and the attraction of the zeros lies just on the root locus.