

# BIOLOGICAL CONTROL SYSTEMS

## THE CONCEPT OF STABILITY AND ROOT LOCUS TECHNIQUE

### THE CONCEPT OF STABILITY

When a system is unstable, the output of the system may be infinite even though the input to the system was finite. This causes a number of practical problems. For instance, a robot arm controller that is unstable may cause the robot to move dangerously. Also, systems that are unstable often incur a certain amount of physical damage, which can become costly. Nonetheless, many systems are inherently unstable - a fighter jet, for instance, or a rocket at liftoff, are examples of naturally unstable systems. Although we can design controllers that stabilize the system, it is first important to understand what stability is, how it is determined, and why it matters.

The chapters in this section are heavily mathematical, and many require a background in linear differential equations. Readers without a strong mathematical background might want to review the necessary chapters in the Calculus and Ordinary Differential Equations books (or equivalent) before reading this material.

For most of this chapter we will be assuming that the system is linear, and can be represented either by a set of transfer functions or in state space. Linear systems have an associated characteristic polynomial, and this polynomial tells us a great deal about the stability of the system. Negativeness of any coefficient of a characteristic polynomial indicates that the system is either unstable or at most marginally stable. If any coefficient is zero/negative then we can say that the system is unstable. It is important to note, though, that even if all of the coefficients of the characteristic polynomial are positive the system may still be unstable. We will look into this in more detail below.

A system is defined to be **BIBO Stable** if every bounded input to the system results in a bounded output over the time interval  $[t_0, \infty)$ . This must hold for all initial times  $t_0$ . So long as we don't input infinity to our system, we won't get infinity output.

A system is defined to be **uniformly BIBO Stable** if there exists a positive constant  $k$  that is independent of  $t_0$  such that for all  $t_0$  the following conditions:

$$\|u(t)\| \leq 1$$

$$t \geq t_0$$

implies that

$$\|y(t)\| \leq k$$

There are a number of different types of stability, and keywords that are used with the topic of stability. Some of the important words that we are going to be discussing in this chapter, and the

## BIOLOGICAL CONTROL SYSTEMS

next few chapters are: **BIBO Stable**, **Marginally Stable**, **Conditionally Stable**, **Uniformly Stable**, **Asymptotically Stable**, and **Unstable**. All of these words mean slightly different things.

Consider the system:

$$h(t) = \frac{2}{t}$$

We can apply our test, selecting an arbitrarily large finite constant  $M$ , and an arbitrary input  $x$  such that  $-M < x < M$ .

As  $M$  approaches infinity (but does not reach infinity), we can show that:

$$y_{-M} = \lim_{M \rightarrow \infty} \frac{2}{-M} = 0^-$$

And:

$$y_M = \lim_{M \rightarrow \infty} \frac{2}{M} = 0^+$$

So now, we can write out our inequality:

$$y_{-M} \leq y_x \leq y_M$$

$$0^- \leq x < 0^+$$

And this inequality should be satisfied for all possible values of  $x$ . However, we can see that when  $x$  is zero, we have the following:

$$y_x = \lim_{x \rightarrow 0} \frac{2}{x} = \infty$$

Which means that  $x$  is between  $-M$  and  $M$ , but the value  $y_x$  is not between  $y_{-M}$  and  $y_M$ . Therefore, this system is not stable.

### Poles and Stability

When the poles of the closed-loop transfer function of a given system are located in the right-half of the  $S$ -plane (RHP), the system becomes unstable. When the poles of the system are located in the left-half plane (LHP) and the system is not improper, the system is shown to be stable. A number of tests deal with this particular facet of stability: The **Routh-Hurwitz Criteria**, the **Root-Locus**, and the **Nyquist Stability Criteria** all test whether there are poles of the transfer function in the RHP. We will learn about all these tests in the upcoming chapters.

If the system is a multivariable, or a MIMO system, then the system is stable if and only if *every pole of every transfer function* in the transfer function matrix has a negative real part and every transfer function in the transfer function matrix is not improper. For these systems, it is possible to use the Routh-Hurwitz, Root Locus, and Nyquist methods described later, but these methods must be performed once for each individual transfer function in the transfer function matrix.

## BIOLOGICAL CONTROL SYSTEMS

Let us remember our generalized feedback-loop transfer function, with a gain element of  $K$ , a forward path  $Gp(s)$ , and a feedback of  $Gb(s)$ . We write the transfer function for this system as:

$$H_{cl}(s) = \frac{KGp(s)}{1 + H_{ol}(s)}$$

Where  $H_{cl}$  is the closed-loop transfer function, and  $H_{ol}$  is the open-loop transfer function. Again, we define the open-loop transfer function as the product of the forward path and the feedback elements, as such:

$$H_{ol}(s) = KGp(s)Gb(s) \text{ <---Note this definition now contradicts the updated definition in the "Feedback" section.}$$

Now, we can define  $F(s)$  to be the **characteristic equation**.  $F(s)$  is simply the denominator of the closed-loop transfer function, and can be defined as such:

**[Characteristic Equation]**

$$F(s) = 1 + H_{ol} = D(s)$$

We can say conclusively that the roots of the characteristic equation are the poles of the transfer function. Now, we know a few simple facts:

1. The locations of the poles of the closed-loop transfer function determine if the system is stable or not
2. The zeros of the characteristic equation are the poles of the closed-loop transfer function.
3. The characteristic equation is always a simpler equation than the closed-loop transfer function.

These functions combined show us that we can focus our attention on the characteristic equation, and find the roots of that equation.

### State-Space and Stability

As we have discussed earlier, the system is stable if the eigenvalues of the system matrix  $A$  have negative real parts. However, there are other stability issues that we can analyze, such as whether a system is *uniformly stable*, *asymptotically stable*, or otherwise. We will discuss all these topics in a later chapter.

### Marginal Stability

When the poles of the system in the complex  $S$ -Domain exist on the complex frequency axis (the vertical axis), or when the eigenvalues of the system matrix are imaginary (no real part), the system exhibits oscillatory characteristics, and is said to be marginally stable. A marginally stable system may become unstable under certain circumstances, and may be perfectly stable under other circumstances.

## ROUTH STABILITY CRITERION:

The Routh approximation method which has been suggested for the reduction of stable discrete time linear systems to guarantee stable models, uses the bilinear transformation. A stability theorem in the  $z$ -plane is presented which is shown to be an equivalent of the Routh criterion. An efficient method that avoids the bilinear transformation is presented by which the Routhdiscrete models are derived directly in the  $z$ -plane.

In control system theory, the **Routh–Hurwitz stability criterion** is a mathematical test that is a necessary and sufficient condition for the stability of a linear time invariant (LTI) control system. The Routh test is an efficient recursive algorithm that English mathematician Edward John Routh proposed in 1876 to determine whether all the roots of the characteristic polynomial of a linear system have negative real parts.<sup>[1]</sup> German mathematician Adolf Hurwitz independently proposed in 1895 to arrange the coefficients of the polynomial into a square matrix, called the Hurwitz matrix, and showed that the polynomial is stable if and only if the sequence of determinants of its principal submatrices are all positive.<sup>[2]</sup> The two procedures are equivalent, with the Routh test providing a more efficient way to compute the Hurwitz determinants than computing them directly. A polynomial satisfying the Routh–Hurwitz criterion is called a Hurwitz polynomial.

The importance of the criterion is that the roots  $p$  of the characteristic equation of a linear system with negative real parts represent solutions  $e^{pt}$  of the system that are stable (bounded). Thus the criterion provides a way to determine if the equations of motion of a linear system have only stable solutions, without solving the system directly. For discrete systems, the corresponding stability test can be handled by the Schur–Cohn criterion, the Jury test and the Bistritz test. With the advent of computers, the criterion has become less widely used, as an alternative is to solve the polynomial numerically, obtaining approximations to the roots directly.

The Routh test can be derived through the use of the Euclidean algorithm and Sturm's theorem in evaluating Cauchy indices. Hurwitz derived his conditions differently.

Using Euclid's algorithm

The criterion is related to Routh–Hurwitz theorem. Indeed, from the statement of that theorem, we have  $p - q = w(+\infty) - w(-\infty)$  where:

- $p$  is the number of roots of the polynomial  $f(z)$  with negative Real Part;
- $q$  is the number of roots of the polynomial  $f(z)$  with positive Real Part (let us remind ourselves that  $f$  is supposed to have no roots lying on the imaginary line);
- $w(x)$  is the number of variations of the generalized Sturm chain obtained from  $P_0(y)$  and  $P_1(y)$  (by successive Euclidean divisions) where  $f(iy) = P_0(y) + iP_1(y)$  for a real  $y$ .

By the fundamental theorem of algebra, each polynomial of degree  $n$  must have  $n$  roots in the complex plane (i.e., for an  $f$  with no roots on the imaginary line,  $p + q = n$ ). Thus, we have the

## BIOLOGICAL CONTROL SYSTEMS

condition that  $f$  is a (Hurwitz) stable polynomial if and only if  $p - q = n$  (the proof is given below). Using the Routh–Hurwitz theorem, we can replace the condition on  $p$  and  $q$  by a condition on the generalized Sturm chain, which will give in turn a condition on the coefficients of  $f$ .

Using matrices

Let  $f(z)$  be a complex polynomial. The process is as follows:

1. Compute the polynomials  $P_0(y)$  and  $P_1(y)$  such that  $f(iy) = P_0(y) + iP_1(y)$  where  $y$  is a real number.
2. Compute the Sylvester matrix associated to  $P_0(y)$  and  $P_1(y)$ .
3. Rearrange each row in such a way that an odd row and the following one have the same number of leading zeros.
4. Compute each principal minor of that matrix.
5. If at least one of the minors is negative (or zero), then the polynomial  $f$  is not stable.

### Example

- Let  $f(z) = az^2 + bz + c$  (for the sake of simplicity we take real coefficients) where  $c \neq 0$  (to avoid a root in zero so that we can use the Routh–Hurwitz theorem). First, we have to calculate the real polynomials  $P_0(y)$  and  $P_1(y)$ :

$$f(iy) = -ay^2 + iby + c = P_0(y) + iP_1(y) = -ay^2 + c + i(by).$$

Next, we divide those polynomials to obtain the generalized Sturm chain:

- $P_0(y) = ((-a/b)y)P_1(y) + c$ , yields  $P_2(y) = -c$ ,
- $P_1(y) = ((-b/c)y)P_2(y)$ , yields  $P_3(y) = 0$  and the Euclidean division stops.

Notice that we had to suppose  $b$  different from zero in the first division. The generalized Sturm chain is in this case  $(P_0(y), P_1(y), P_2(y)) = (c - ay^2, by, -c)$ . Putting  $y = +\infty$ , the sign of  $c - ay^2$  is the opposite sign of  $a$  and the sign of  $by$  is the sign of  $b$ . When we put  $y = -\infty$ , the sign of the first element of the chain is again the opposite sign of  $a$  and the sign of  $by$  is the opposite sign of  $b$ . Finally,  $-c$  has always the opposite sign of  $c$ .

Suppose now that  $f$  is Hurwitz-stable. This means that  $w(+\infty) - w(-\infty) = 2$  (the degree of  $f$ ). By the properties of the function  $w$ , this is the same as  $w(+\infty) = 2$  and  $w(-\infty) = 0$ . Thus,  $a$ ,  $b$  and  $c$  must have the same sign. We have thus found the necessary condition of stability for polynomials of degree 2.