Poles and zeroes

Log of the absolute value of the gain of an 8th order Chebyshev type II filter in complex frequency space (s=σ+jω) with ε = 0.1 and $\omega_0 = 1$. The white spots are poles and the black spots are zeroes. All 16 poles are shown. Each zero has multiplicity of two, and 12 zeroes are shown and four are located outside the picture, two on the positive ω axis, and two on the negative. The poles of the transfer function are poles on the left half plane and the zeroes of the transfer function are the zeroes, but with multiplicity 1. Black corresponds to a gain of 0.05 or less, white corresponds to a gain of 20 or more.

Assuming that the cutoff frequency is equal to unity, the poles (ω_{pm}) of the gain of the Chebyshev filter are the zeroes of the denominator of the gain:

$$1 + \varepsilon^2 T_n^2(-1/js_{pm}) = 0.$$

The poles of gain of the type II Chebyshev filter are the inverse of the poles of the type I filter:

$$\frac{1}{s_{pm}^{\pm}} = \pm \sinh\left(\frac{1}{n}\mathrm{arsinh}\left(\frac{1}{\varepsilon}\right)\right)\sin(\theta_m)$$
$$+j\cosh\left(\frac{1}{n}\mathrm{arsinh}\left(\frac{1}{\varepsilon}\right)\right)\cos(\theta_m)$$

where $m = 1, 2, ..., n$. The zeroes (ω_{zm}) of the type II Chebyshev filter are the zeroes of the numerator of the gain:

$$\varepsilon^2 T_n^2(-1/js_{zm}) = 0.$$

The zeroes of the type II Chebyshev filter are therefore the inverse of the zeroes of the Chebyshev polynomial.

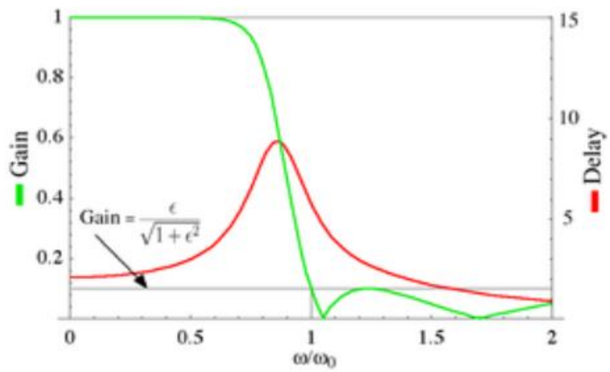
$$1/s_{zm} = -j\cos\left(\frac{\pi}{2}\,\frac{2m-1}{n}\right)$$

for $m = 1, 2, ..., n$.

The transfer function

The transfer function is given by the poles in the left half plane of the gain function, and has the same zeroes but these zeroes are single rather than double zeroes.

The group delay



Gain and group delay of a fifth-order type II Chebyshev filter with ε = 0.1.

The gain and the group delay for a fifth-order type II Chebyshev filter with ε=0.1 are plotted in the graph on the left. It can be seen that there are ripples in the gain in the stopband but not in the pass band.

Implementation

Cauer topology

A passive LC Chebyshev low-pass filter may be realized using a Cauer topology. The inductor or capacitor values of a nth-order Chebyshev prototype filter may be calculated from the following equations:[1]

$$G_0 = 1$$
$$G_1 = \frac{2A_1}{\gamma}$$
$$G_k = \frac{4A_{k-1}A_k}{B_{k-1}G_{k-1}}, \qquad k = 2,3,4,\dots,n$$
$$G_{n+1} = \begin{cases} 1 & \text{if } n \text{ odd} \\ \coth^2\left(\frac{\beta}{4}\right) & \text{if } n \text{ even} \end{cases}$$

G_1, G_k are the capacitor or inductor element values. f_H, the 3 dB frequency is calculated with:

$$f_H = f_0 \cosh\left(\frac{1}{n}\cosh^{-1}\frac{1}{\varepsilon}\right)$$

The coefficients A, γ, β, A_k, and B_k may be calculated from the following equations:

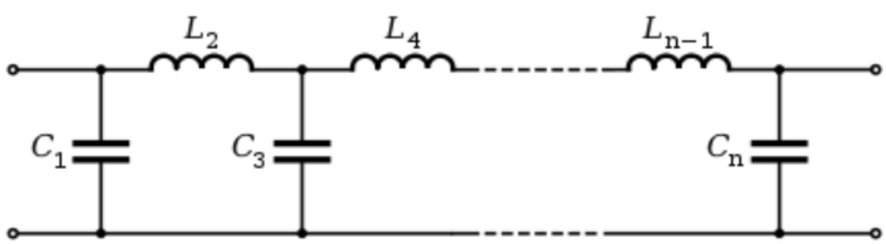
$$\gamma = \sinh\left(\frac{\beta}{2n}\right)$$

$$\beta = \ln\left[\coth\left(\frac{R_{db}}{17.37}\right)\right]$$

$$A_k = \sin\frac{(2k-1)\pi}{2n}, \qquad k = 1, 2, 3, \ldots, n$$

$$B_k = \gamma^2 + \sin^2\left(\frac{k\pi}{n}\right), \qquad k = 1, 2, 3, \ldots, n$$

where R_{dB} is the passband ripple in decibels.



Low-pass filter using Cauer topology

The calculated G_k values may then be converted into shunt capacitors and series inductors as shown on the right, or they may be converted into series capacitors and shunt inductors. For example,

- $C_{1\ \text{shunt}} = G_1$, $L_{2\ \text{series}} = G_2$, ...

or

- $L_{1\ \text{shunt}} = G_1$, $C_{1\ \text{series}} = G_2$, ...

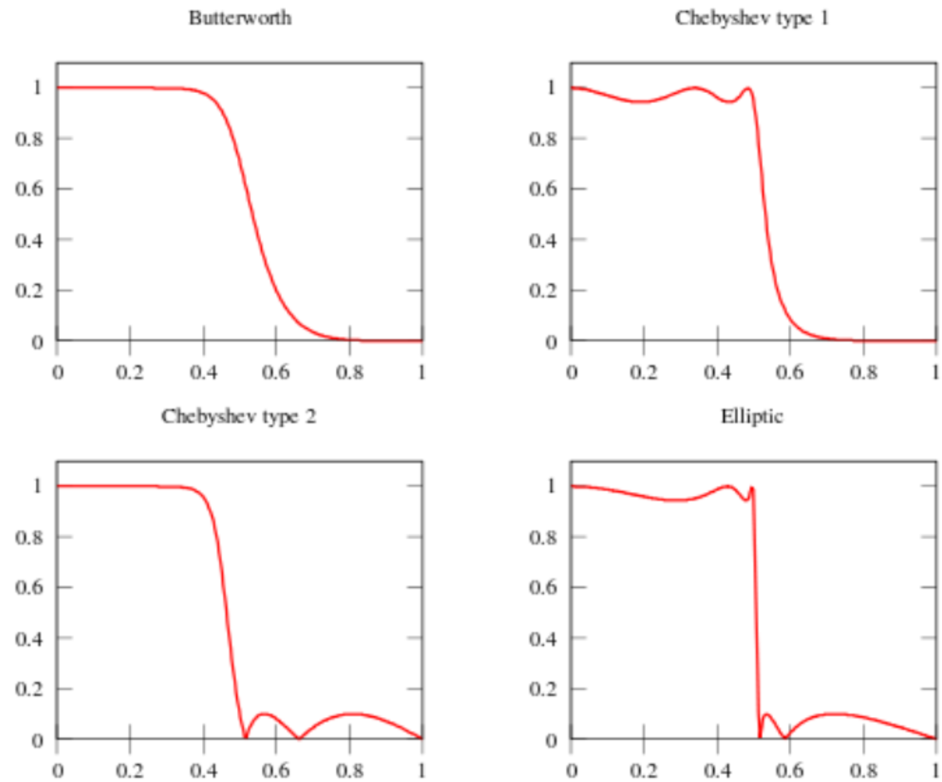
Note that when G_1 is a shunt capacitor or series inductor, G_0 corresponds to the input resistance or conductance, respectively. The same relationship holds for G_{n+1} and G_n. The resulting circuit is a normalized low-pass filter. Using frequency transformations and impedance scaling, the normalized low-pass filter may be transformed into high-pass, band-pass, and band-stop filters of any desired cutoff frequency or bandwidth.

Digital

As with most analog filters, the Chebyshev may be converted to a digital (discrete-time) recursive form via the bilinear transform. However, as digital filters have a finite bandwidth, the response shape of the transformed Chebyshev is warped. Alternatively, the Matched Z-transform method may be used, which does not warp the response.

Comparison with other linear filters

The following illustration shows the Chebyshev filters next to other common filter types obtained with the same number of coefficients (fifth order):



Chebyshev filters are sharper than the Butterworth filter; they are not as sharp as the elliptic one, but they show fewer ripples over the bandwidth.

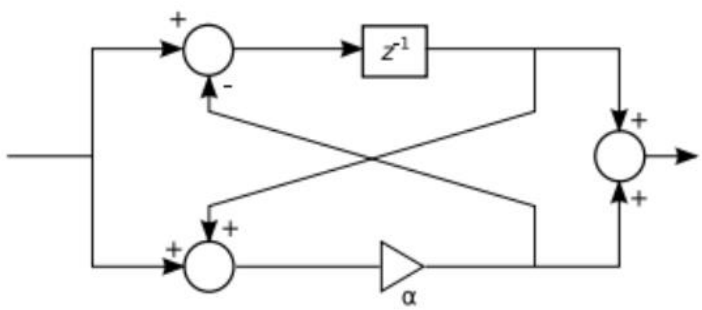
frequency transformation in analog and digital domain

There are two approaches to the design of digital filters of bandpass, highpass, and bandstop types. Approach 1: Design an analog lowpass filter, apply the frequency band transformations in analog domain, and then map the relevant filter to a digital filter Disadvantage Due to aliasing problem inherent in the use of impulse invariant technique a bandpass or highpass filter cannot be transformed. Approach 2: Design an analog lowpass filter, map it to a digital filter, and then apply frequency band transformations in digital domain to obtain the desired digital filter. In this handout we introduce the second approach by defining several transformations to map a LPF to a LPF, BPF, and HPF with given specifications. Define a mapping from the z-plane to the ˜z-plane of the form $z^{-1} = f(\tilde{z}^{-1})$ (1) such that the transfer function $H(z^{-1}) \xrightarrow{\text{mapped}} G(\tilde{z}^{-1})$ (2) Conditions for this mapping are a $f(\cdot)$ is real and rational. b $f(\cdot)$ must produce stable $G(\tilde{z}^{-1})$ from stable $H(z^{-1})$ i.e. the interior of the unit circle in the z-plane must be mapped to the interior of the unit circle in the ˜z-plane. c $|f(\tilde{z}^{-1})| = 1$, $|z^{-1}| = 1$ d The inverse mapping exists i.e. ˜z −1 = f −1 (z −1) A class of transformation for mapping a LPF to another LPF with different frequency characteristics or to a HPF, has a general form of f(˜z −1) = a0 + a1z˜ −1 1 + b1z˜ −1

Lowpass to Lowpass Mapping

The transformation

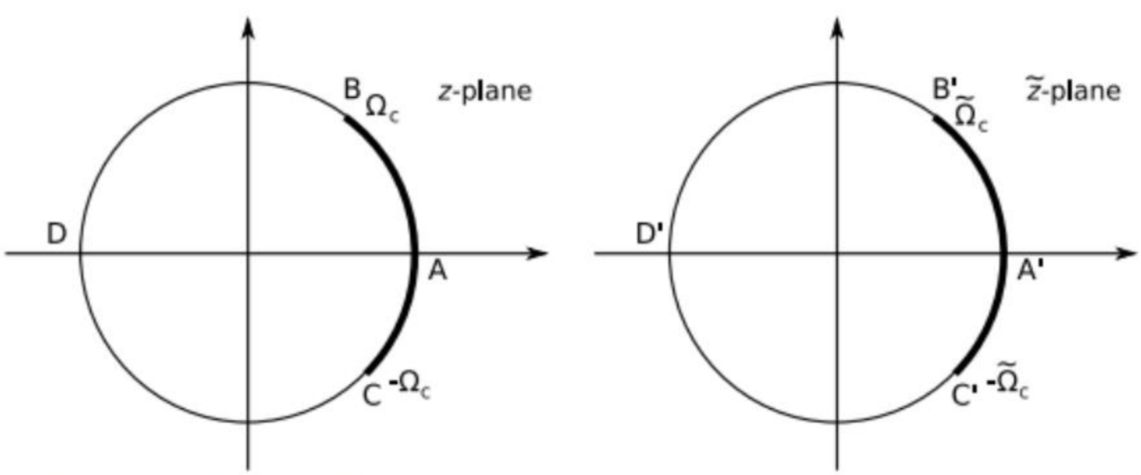
$$z^{-1} = f(\tilde{z}^{-1}) = \frac{\tilde{z}^{-1} - \alpha}{1 - \alpha\tilde{z}^{-1}}$$



which satisfies the above conditions can be used to map a LPF to another LPF. Let $z = e^{j\Omega T}$ and $\tilde{z} = e^{j\tilde{\Omega}T}$. Then the relationship between the frequencies Ω and $\tilde{\Omega}$ is

$$\Omega = \frac{2}{T}\tan^{-1}(K \tan\frac{\tilde{\Omega}T}{2}),\ K = \frac{1+\alpha}{1-\alpha}$$

where α can be obtained based upon the cutoff frequency of the transformed filter.

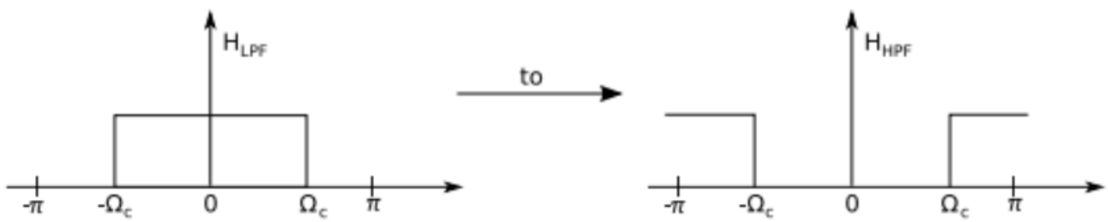


(a) Passband and stopband of the original LPF on the unit circle

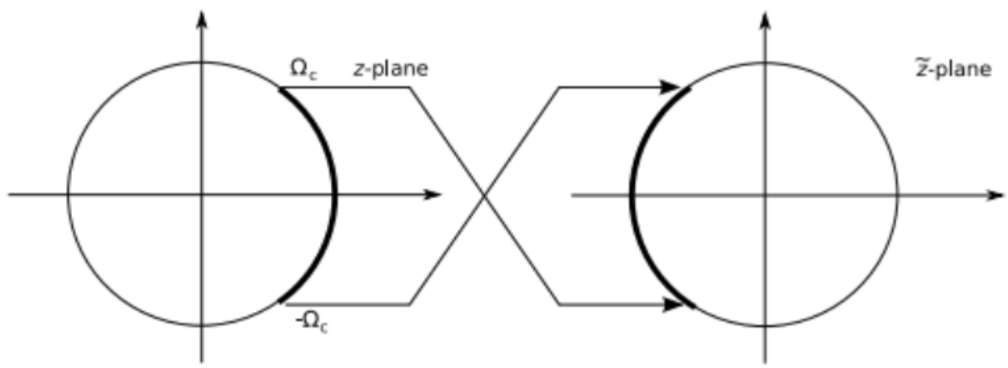
(b) Passband and stopband of the mapped LPF on the unit circle

Lowpass to Highpass Mapping

We desire to map



On the unit circle we must rotate the frequency band.



Thus, the mapping is

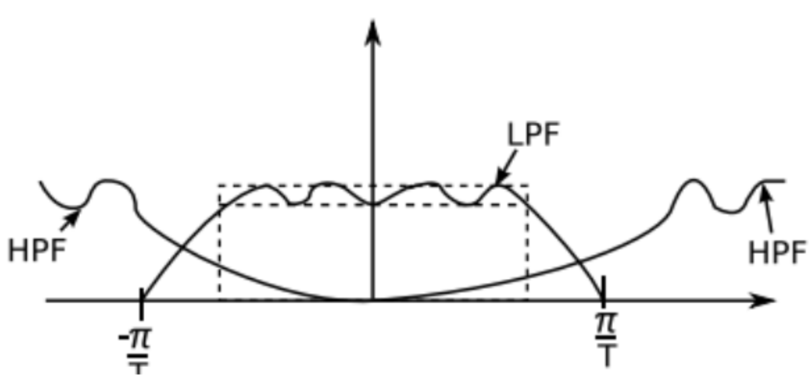
$$z^{-1} \longrightarrow -\tilde{z}^{-1},$$

or in general

$$z^{-1} \longrightarrow \frac{-(\tilde{z}^{-1}+\alpha)}{1+\alpha\tilde{z}^{-1}}$$

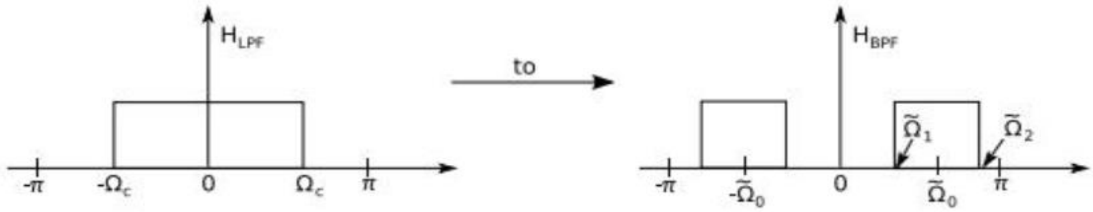
The cutoff frequency of the HPF is related to that of LPF by

$$\Omega_{c_{HP}} + \Omega_{c_{LP}} = \frac{\pi}{T} \text{ (normalized case)}$$



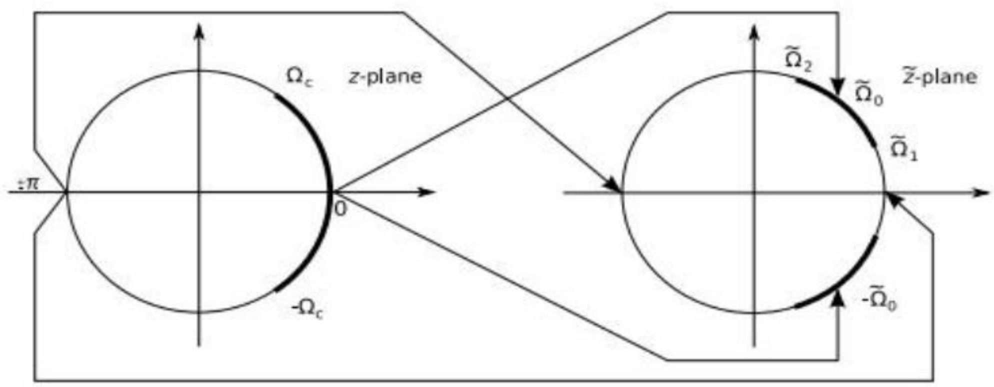
Lowpass to Bandpass Mapping

We desire to map



Note that since every point in LPF characteristics is mapped to two points in that of the BPF, we need a 2nd order mapping i.e.

$$z^{-1} = f(\tilde{z}^{-1}) = \frac{a_0 + a_1\tilde{z}^{-1} + a_2\tilde{z}^{-2}}{b_0 + b_1\tilde{z}^{-1} + b_2\tilde{z}^{-2}}$$



It can be shown that the mapping which satisfies the above mentioned conditions will have a form

$$z^{-1} = f(\tilde{z}^{-1}) = -\left[\frac{a_0 + a_1\tilde{z}^{-1} + a_2\tilde{z}^{-2}}{a_2 + a_1\tilde{z}^{-1} + a_0\tilde{z}^{-2}}\right]$$

or in a more useful form

$$z^{-1} = f(\tilde{z}^{-1}) = -\left[\frac{\tilde{z}^{-2} - \frac{2\alpha K}{K+1}\tilde{z}^{-1} + \frac{K-1}{K+1}}{\frac{K-1}{K+1}\tilde{z}^{-2} - \frac{2\alpha K}{K+1}\tilde{z}^{-1} + 1}\right]$$

where

$$\begin{aligned} K &= \frac{a_0 + a_2}{a_0 - a_2} \\ &= \tan\frac{\Omega_c T}{2}\left[\cot\left[\left(\frac{\tilde{\Omega}_2 - \tilde{\Omega}_1}{2}\right)T\right]\right] \end{aligned} \qquad (12)$$

and $\alpha = \cos\tilde{\Omega}_0$, $T = -a_1/(a_0 + a_2)$, and $\tan\left(\frac{(\tilde{\Omega}_1 - \tilde{\Omega}_2)T}{2}\right) = \tan\left(\frac{\Omega_c T}{2}\right)$