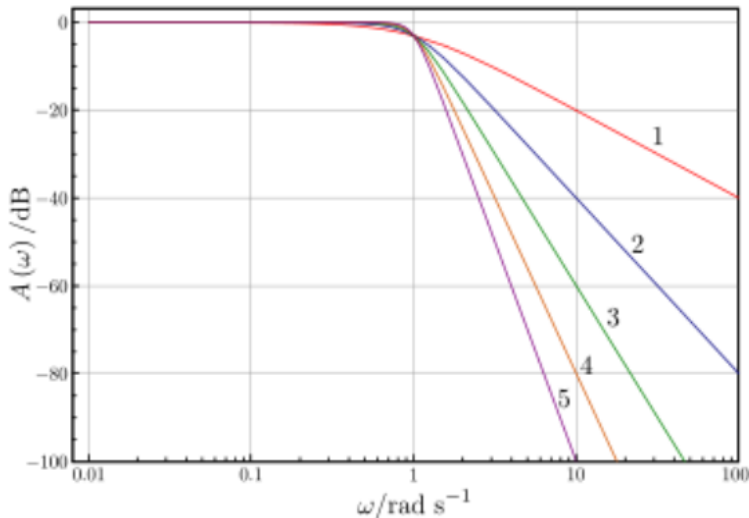


IIR FILTER DESIGN

BUTTERWORTH AND CHEBYSHEV FILTERS CONT'D

Transfer function



Plot of the gain of Butterworth low-pass filters of orders 1 through 5, with cutoff frequency $\omega_0 = 1$. Note that the slope is $20n$ dB/decade where n is the filter order.

Like all filters, the typical prototype is the low-pass filter, which can be modified into a high-pass filter, or placed in series with others to form band-pass and band-stop filters, and higher order versions of these.

The gain $G(\omega)$ of an n -order Butterworth low pass filter is given in terms of the transfer function $H(s)$ as

$$G^2(\omega) = |H(j\omega)|^2 = \frac{G_0^2}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}$$

where

- n = order of filter
- ω_c = cutoff frequency (approximately the -3dB frequency)
- G_0 is the DC gain (gain at zero frequency)

It can be seen that as n approaches infinity, the gain becomes a rectangle function and frequencies below ω_c will be passed with gain G_0 , while frequencies above ω_c will be suppressed. For smaller values of n , the cutoff will be less sharp.

We wish to determine the transfer function $H(s)$ where $s = \sigma + j\omega$ (from Laplace transform). Because $|H(s)|^2 = H(s)\overline{H(s)}$ and, as a general property of Laplace transforms at $s = j\omega$, $H(-j\omega) = \overline{H(j\omega)}$, if we select $H(s)$ such that:

$$H(s)H(-s) = \frac{G_0^2}{1 + \left(\frac{-s^2}{\omega_c^2}\right)^n},$$

then, for imaginary inputs, $s = j\omega$, we have the frequency response of the Butterworth filter.

The n poles of this expression occur on a circle of radius ω_c at equally-spaced points, and symmetric around the imaginary axis. For stability, the transfer function, $H(s)$, is therefore chosen such that it contains only the poles in the negative real half-plane of s . The k -th pole is specified by

$$-\frac{s_k^2}{\omega_c^2} = (-1)^{\frac{1}{n}} = e^{\frac{j(2k-1)\pi}{n}} \quad k = 1, 2, 3, \dots, n$$

and hence;

$$s_k = \omega_c e^{\frac{j(2k+n-1)\pi}{2n}} \quad k = 1, 2, 3, \dots, n.$$

The transfer(or system) function may be written in terms of these poles as

$$H(s) = \frac{G_0}{\prod_{k=1}^n (s - s_k) / \omega_c}.$$

The denominator is a Butterworth polynomial in s .

Normalized Butterworth polynomials

The Butterworth polynomials may be written in complex form as above, but are usually written with real coefficients by multiplying that pole pairs that are complex conjugates, such as s_1 and s_n . The polynomials are normalized by setting $\omega_c = 1$. The normalized Butterworth polynomials then have the general form

$$B_n(s) = \prod_{k=1}^{\frac{n}{2}} \left[s^2 - 2s \cos \left(\frac{2k + n - 1}{2n} \pi \right) + 1 \right] \quad n = \text{even}$$

$$B_n(s) = (s + 1) \prod_{k=1}^{\frac{n-1}{2}} \left[s^2 - 2s \cos \left(\frac{2k + n - 1}{2n} \pi \right) + 1 \right] \quad n = \text{odd.}$$

To four decimal places, they are

n Factors of Polynomial $B_n(s)$

$$1 (s + 1)$$

$$2 s^2 + 1.4142s + 1$$

$$3 (s + 1)(s^2 + s + 1)$$

$$4 (s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)$$

$$5 (s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$$

$$6 (s^2 + 0.5176s + 1)(s^2 + 1.4142s + 1)(s^2 + 1.9319s + 1)$$

$$7 (s + 1)(s^2 + 0.4450s + 1)(s^2 + 1.2470s + 1)(s^2 + 1.8019s + 1)$$

$$8 (s^2 + 0.3902s + 1)(s^2 + 1.1111s + 1)(s^2 + 1.6629s + 1)(s^2 + 1.9616s + 1)$$

The normalized Butterworth polynomials can be used to determine the transfer function for any low-pass filter cut-off frequency ω_c , as follows

$$H(s) = \frac{G_0}{B_n(a)}, \text{ where } a = \frac{s}{\omega_c}.$$

Transformation to other bandforms are also possible, see prototype filter.

Maximal flatness

Assuming $\omega_c = 1$ and $G_0 = 1$, the derivative of the gain with respect to frequency can be shown to be

$$\frac{dG}{d\omega} = -nG^3\omega^{2n-1}$$

which is monotonically decreasing for all ω since the gain G is always positive. The gain function of the Butterworth filter therefore has no ripple. Furthermore, the series expansion of the gain is given by

$$G(\omega) = 1 - \frac{1}{2}\omega^{2n} + \frac{3}{8}\omega^{4n} + \dots$$

In other words, all derivatives of the gain up to but not including the $2n$ -th derivative are zero at $\omega = 0$, resulting in "maximal flatness". If the requirement to be monotonic is limited to the passband only and ripples are allowed in the stopband, then it is possible to design a filter of the same order, such as the inverse Chebyshev filter, that is flatter in the passband than the "maximally flat" Butterworth.

High-frequency roll-off

Again assuming $\omega_c = 1$, the slope of the log of the gain for large ω is

$$\lim_{\omega \rightarrow \infty} \frac{d \log(G)}{d \log(\omega)} = -n.$$

In decibels, the high-frequency roll-off is therefore $20n$ dB/decade, or $6n$ dB/octave (the factor of 20 is used because the power is proportional to the square of the voltage gain; see 20 log rule.)