Continued fractions: basic definitions

Let θ be any real number. Put $a_0 = \lfloor\theta\rfloor$ (the largest integer not greater than θ). If $a_0 \neq \theta$, then we can write $\theta = a_0 + \frac{1}{\theta_1}$, where $\theta_1 > 1$, and we put $a_1 = \lfloor\theta_1\rfloor$. If $a_1 \neq \theta_1$, then we can write $\theta_1 = a_1 + \frac{1}{\theta_2}$, where $\theta_2 > 1$, and we put $a_2 = \lfloor\theta_2\rfloor$. This process can be continued indefinitely, unless $a_n = \theta_n$ for some n. Note that $a_1, a_2, \ldots$ are all positive integers, although a_0 might be negative or zero. This process is the **continued fraction process**, and the a_i are known as the **partial quotients** of θ.

If the process terminates, then we have

$$\begin{aligned}\theta &= a_0 + \frac{1}{\theta_1} \\ &= a_0 + \frac{1}{a_1 + \frac{1}{\theta_2}} \\ &\vdots \\ &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}}\end{aligned}$$

We then write

$$\theta = [a_0, a_1, \ldots, a_n].$$

We also use this notation when the a_i are not necessarily integers.

If the continued fraction process does not terminate, then we write

$$\theta = [a_0, a_1, a_2, \ldots],$$

and for any n we then have

$$\theta = [a_0, a_1, a_2, \ldots, a_n, \theta_{n+1}],$$

where $a_0, \ldots, a_n$ are integers, but θ_{n+1} is not.

If we set

$$\frac{p_n}{q_n} = [a_0, \ldots, a_n],$$

where $\gcd(p_n, q_n) = 1$, then we call $\frac{p_n}{q_n}$ the nth **convergent** to θ. We shall see that

$$\frac{p_n}{q_n} \to \theta \text{ as } n \to \infty.$$

Continued fractions: a recurrence relation for the convergents

Let $a_0, a_1, a_2, \ldots$ be a sequence of integers, with $a_i > 0$ when $i > 0$. Define p_n, q_n by

$$p_0 = a_0,\ q_0 = 1,\ p_1 = a_0 a_1 + 1,\ q_1 = a_1,$$

$$p_n = a_n p_{n-1} + p_{n-2},\ q_n = a_n q_{n-1} + q_{n-2},\ \text{for } n \geq 2.$$

Then:

(a) $p_n q_{n+1} - p_{n+1} q_n = (-1)^{n+1}$;

(b) $\gcd(p_n, q_n) = 1$;

(c) $\frac{p_n}{q_n} = [a_0, \ldots, a_n]$;

(d) If the a_i are produced by applying the continued fraction process to θ, then $\frac{p_n}{q_n}$ is the nth convergent to θ, and

$$\theta = \frac{p_n \theta_{n+1} + p_{n-1}}{q_n \theta_{n+1} + q_{n-1}}.$$

Proof
(a) We use induction on n. We have $p_0q_1 - p_1q_0 = a_0a_1 - a_0a_1 - 1 = -1$, so the result holds for $n = 0$. Suppose that the result holds for $n = m - 1$, and consider the case $n = m$. We have, using the recurrence relation,

$$\begin{aligned} p_m q_{m+1} - p_{m+1} q_m &= p_m(a_{m+1}q_m + q_{m-1}) - (a_{m+1}p_m + p_{m-1})q_m \\ &= p_m q_{m-1} - p_{m-1} q_m \\ &= -(-1)^m \\ &= (-1)^{m+1}, \end{aligned}$$

so the result holds for $n = m$.
(b) This is immediate from (a).
The remark in (d) that p_n/q_n is the nth convergent to θ follows immediately from (c). We use induction on n to prove the rest of (d) along with (c), remembering that (c) does not require *a priori* that the a_i are produced by the continued fraction process. First note that

$$\frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = [a_0, a_1].$$

Also

$$\begin{aligned} \frac{p_1\theta_2 + p_0}{q_1\theta_2 + q_0} &= \frac{(a_0a_1 + 1)\theta_2 + a_0}{a_1\theta_2 + 1} \\ &= a_0 + \frac{\theta_2}{a_1\theta_2 + 1} \\ &= a_0 + \frac{1}{a_1 + \frac{1}{\theta_2}} \\ &= \theta, \end{aligned}$$

so the result holds for $n = 1$. Suppose that the result holds for $n = m - 1$, and consider the case $n = m$. We have

$$\begin{aligned}[a_0,\ldots,a_m] &= \frac{p_{m-1}a_m+p_{m-2}}{q_{m-1}a_m+q_{m-2}},\ \text{using (d), with } n=m-1\\ &\qquad\qquad\qquad \text{and } \theta=[a_0,\ldots,a_m]\\ &= \frac{p_m}{q_m},\ \text{which is (c), with } n=m.\end{aligned}$$

To establish (d) with $n = m$, note that

$$\begin{aligned}\theta &= [a_0,\ldots,a_m,\theta_{m+1}]\\ &= [a_0,\ldots,a_{m-1},a_m+\frac{1}{\theta_{m+1}}]\\ &= \frac{p_{m-1}(a_m+\frac{1}{\theta_{m+1}})+p_{m-2}}{q_{m-1}(a_m+\frac{1}{\theta_{m+1}})+q_{m-2}},\ \text{using (d), with } n=m-1\\ &= \frac{p_m+\frac{p_{m-1}}{\theta_{m+1}}}{q_m+\frac{q_{m-1}}{\theta_{m+1}}},\ \text{using the recurrence relations}\\ &= \frac{p_m\theta_{m+1}+p_{m-1}}{q_m\theta_{m+1}+q_{m-1}},\ \text{which is (d), with } n=m.\end{aligned}$$

Continued fractions: some properties of the convergents

For parts (a) to (d), we suppose that the continued fraction process does not terminate.

(a) θ lies between $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$.
Proof $\theta = [a_0,\ldots,a_n,\theta_{n+1}] = [a_0,\ldots,a_n+\frac{1}{\theta_{n+1}}]$, where $0 < \frac{1}{\theta_{n+1}} \le \frac{1}{a_{n+1}}$, so θ lies between $[a_0,\ldots,a_n]$ and $[a_0,\ldots,a_n+\frac{1}{a_{n+1}}] = [a_0,\ldots,a_n,a_{n+1}]$.

(b) $|\theta - \frac{p_n}{q_n}| \le \frac{1}{q_nq_{n+1}}$.
Proof From (a), $|\theta - \frac{p_n}{q_n}| \le |\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}| = \frac{1}{q_nq_{n+1}}$, using 1.5(a).

(c) $q_{n+2} \ge 2q_n, p_{n+2} \ge 2p_n \quad (n \ge 1)$
Proof Immediate from the recurrence relations.

(d) $\frac{p_n}{q_n} \to \theta$ as $n \to \infty$.
Proof Immediate from (b) and (c).

(e) The continued fraction process terminates if and only if θ is rational.
Proof The 'only if' part is clear. Conversely, suppose that $\theta = \frac{a}{b}$ is rational, and that the process does not terminate. Then taking n such that $q_{n+1} > b$ gives $|\theta - \frac{p_n}{q_n}| \geq \frac{1}{bq_n} > \frac{1}{q_n q_{n+1}}$, contradicting (b).

Note that 1.5(a) could be used to compute inverses mod n. To compute the inverse of a mod n, we compute convergents to $\frac{a}{n}$. By (e), we eventually reach $p_r = a, q_r = n$, provided that $\gcd(a, n) = 1$. By 1.5(a), we then have $p_{r-1}n - aq_{r-1} = (-1)^{r+1}$, so that q_{r-1} is, up to choice of sign, the desired inverse. This method is equivalent to (a variant of) Euclid's algorithm. From (c), we have $r = O(\log n)$.

The continued fraction process gives us a sequence of rational approximations to any irrational number θ. These approximations are rather good, indeed they are the 'best possible' in a sense made precise below.

Examples

(a)

$$\theta = \tfrac{16}{9}, \quad a_0 = 1, \quad \theta = 1 + \tfrac{7}{9}.$$

$$\theta_1 = \tfrac{9}{7}, \quad a_1 = 1, \quad \theta_1 = 1 + \tfrac{2}{7}.$$

$$\theta_2 = \tfrac{7}{2}, \quad a_2 = 3, \quad \theta_2 = 3 + \tfrac{1}{2}.$$

$$\theta_3 = 2 \quad a_3 = 2 \quad \theta_3 = 2$$

$\frac{16}{9} = [1, 1, 3, 2]$

(Compare with Euclid's algorithm.)

$\frac{p_0}{q_0} = \frac{1}{1}$

$\frac{p_1}{q_1} = 1 + \frac{1}{1} = \frac{2}{1}$

$\frac{p_2}{q_2} = 1 + \frac{1}{1+\frac{1}{3}} = 1 + \frac{3}{4} = \frac{7}{4}$

$\frac{p_3}{q_3} = 1 + \frac{1}{1+\frac{1}{3+\frac{1}{2}}} = 1 + \frac{1}{1+\frac{2}{7}} = 1 + \frac{7}{9} = \frac{16}{9}$

(Check the properties of the convergents proved above. Remark that we have computed the inverse of 16 mod 9, and of 9 mod 16.)

(b) $\theta = \sqrt{19} = [4, 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8, \ldots]$
(For irrational numbers, the partial quotients are often mysterious. Two exceptions are quadratic irrationals, and certain functions of e.)

(c) $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$