LECTURE I: BIFURCATION THEORY

1 Review of Phase Planes

Nonlinear systems come in a variety of forms:

- Nonlinear ordinary differential equations (odes), eg. pendulum equation:

$$\ddot{x} + \frac{g}{L}\sin x = 0.$$

- Nonlinear partial differential equations (pdes), eg. Reaction-Diffusion equation:

$$\frac{\partial x}{\partial t} = \nabla^2 x + x^3;$$

  or the Navier Stokes equations of fluid mechanics:

$$u_t + (\underline{u}\cdot\underline{\nabla})\underline{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{u}$$

$$\underline{u}_t + \rho\nabla\cdot\underline{u} = 0.$$

- Nonlinear maps, eg. Logistic equation of population dynamics:

$$x_{n+1} = \mu x_n(1 - x_n).$$

Their characterising feature is that linear superposition of solutions does not hold.

Consider the 2-D autonomous system:

$$\dot{x} = f(x, y) \tag{1.1}$$

$$\dot{y} = g(x, y),$$

where $(\dot{\cdot}) = \frac{d}{dt}$ and 'autonomous' means having no explicit time variation in (1.1) (such as the forcing term 'cos ωt').

The evolution of an instantaneous state of the system defines a trajectory in 2-D phase-space. The set of such trajectories is the phase portrait in the (x, y)-phase plane. Equilibrium or critical or fixed points are solutions to

$$\dot{x} = 0 = \dot{y}, \qquad \text{i.e. to } f(x_e, y_e) = 0 = g(x_e, y_e) \tag{1.2}$$

Linear stability about an equilibrium state (x_e, y_e) is found by perturbing about (x_e, y_e) by including small terms:

$$x = x_e + u, y = y_e + v,$$

and using Taylor series to expand about (x_e, y_e), neglecting quadratic and higher order terms in u, v:

$$\dot{u} = f(x_e + u, y_e + v) = f(x_e, y_e) + uf_x(x_e, y_e) + vf_y(x_e, y_e) + O(u^2, v^2) \qquad (1.3)$$

$$\dot{v} = g(x_e + u, y_e + v) = g(x_e, y_e) + ug_x(x_e, y_e) + vg_y(x_e, y_e) + O(u^2, v^2). \qquad (1.4)$$

Using (1.2), we obtain

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}}_{=J} \begin{bmatrix} u \\ v \end{bmatrix}$$

or

$$\dot{\underline{u}} = J_e \underline{u}, \qquad (1.5)$$

where J_e is the Jacobian matrix of (1.1) evaluated at (x_e, y_e).

NB: For n-dimensional systems $\dot{\underline{x}} = \underline{f}(\underline{x})$, we have

$$\dot{\underline{u}} = \left[\frac{\partial f_i}{\partial x_j}\right] \underline{u}.$$

Seeking solutions to (1.5) of the form $\underline{u} = e^{\lambda t}\underline{v}$ gives

$$\lambda e^{\lambda t}\underline{v} = J_e e^{\lambda t}\underline{v}$$

or

$$(J_e - \lambda I)\underline{v} = 0.$$

This is an eigenvalue problem for λ and $\underline{v}$. Thus $\underline{u} = e^{\lambda t}\underline{v}$ is a solution to (1.5) iff λ is an eigenvalue of J_e with eigenvector $\underline{v}$.

$$det|J_e - \lambda I| = 0$$

gives the characteristic equation for λ_j.

We have the following cases:

- If $\Re(\lambda_j) < 0, \forall j$, the equilibrium is asymptotically stable.
- If $\Re(\lambda_i) > 0$, for some i, the equilibrium is unstable.
- If $\Re(\lambda_k) = 0$, for some k, the equilibrium is marginally stable.

Recall from Part A Differential Equations the following classification of equilibria:

- if $\lambda_1, \lambda_2 < 0$: we have a stable node;
- if $\lambda_1 < 0 < \lambda_2$: we have a saddle;
- if $\lambda_1, \lambda_2 > 0$: we have an unstable node;
- if $\lambda_{1,2} = \alpha \pm i\beta$: we have a stable focus for $\alpha > 0$;
- if $\lambda_{1,2} = \alpha \pm i\beta$: we have a centre.

1.1 Procedure for nonlinear analysis

- Find all the equilibria of $\dot{\underline{x}} = f(\underline{x})$.
- Determine their linear stability via $\dot{\underline{u}} = J\underline{u}$.
- Evaluate J at each equilibrium $\underline{x}_e$.
- Determine all the eigenvalues.
- Classify each equillibrium eg. as a saddle, node, etc.
- Sketch the phase portraits.

In this course we shall cover both conservative and non-conservative (dissipative) systems.

1.2 Conservative Systems

The classic example of a ’conservative’ system is Newton’s second law of motion:

$$m\ddot{x} = F(x) \tag{1.6}$$

where $F(x)$ is independent of $\dot{x}$ and no explicit t appears. We can write F in terms of the derivative of a potential function:

$$F = -\frac{-dV}{dx}.$$

Multiplying through by $\dot{x}$ and integrating with respect to t we get

$$\frac{1}{2}m(\dot{x})^2 + V(x) = E, \quad \text{constant},$$

where $V(x)$ is identified as the potential energy and E is the total conserved energy.

1.3 Duffing Oscillator

Consider

$$\dot{x} = x - x^3 = -\frac{dV}{dx}.$$

We can rewrite this as a pair of first order odes:

$$\dot{x} = y = F_1, \quad (1.7a)$$

$$\dot{y} = x - x^3 = F_2. \quad (1.7b)$$

Liouville's Theorem of the Conservation of Phase Space gives $\nabla.\underline{F} = 0$.
The equilibrium states are $x = 0$, $y = 0$ and $y = 0$, $x = \pm 1$. Stability is given by :

$$J_e = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{bmatrix}_{x_e},$$

so that the characteristic equation becomes: $\lambda^2 - (1 - 3x_e^2) = 0$.
For $(0, 0)$, we have $\lambda^2 = 1$ so that $\lambda = \pm 1$, ie. a saddle;
while for $(\pm 1, 0)$, $\lambda^2 + 2 = 0$, giving $\lambda = \pm\sqrt{2i}$ ie. a pair of centres.

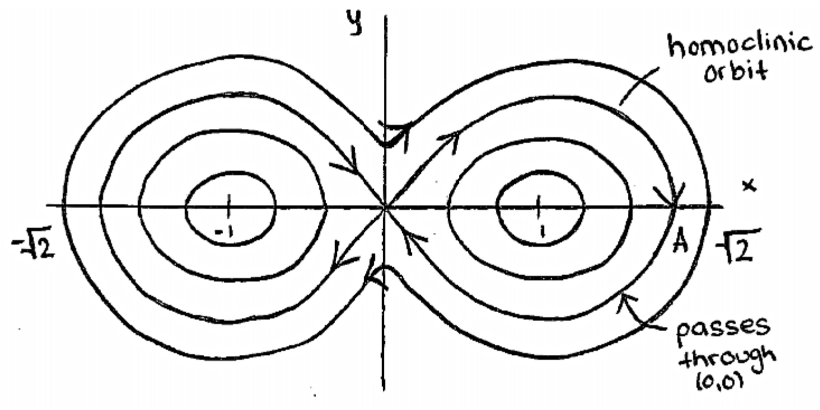
We can obtain the energy equation by taking

$$\dot{x}\dot{y} - \dot{y}\dot{x} = y\dot{y} - x\dot{x} + x^3\dot{x} = 0.$$

Integration with respect to t gives

$$\frac{1}{2}y^2 - \frac{x^2}{2} + \frac{x^4}{4} = E,$$

so that $V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$.
The phase portrait in the (x, y)-plane shows a system of concentric circles, centered about each $(\pm 1, 0)$ equilibrium state, as well as a larger system enveloping all 3 equilibria. The bounding orbit, which connects the saddle point $(0, 0)$ to itself, is called a homoclinic orbit, and is given by $E = 0$



or $y^2 = x^2 - \frac{x^4}{2}$. It takes an infinite length of time to travel around the homoclinic orbit as we now show. Defining

$$I = \oint dt = 2\int_0^A \frac{dt}{dx}dx = 2\int_0^A \frac{dt}{y},$$

where A is given by $x = \sqrt{2}$, and since $\dot{x} = y$, we have $y = x\sqrt{1 - \frac{x^2}{2}}$ on the homoclinic orbit.

Therefore

$$I = 2\int_0^{\sqrt{2}} \frac{dx}{x\sqrt{1-\frac{x^2}{2}}} = 2\int_0^{\sqrt{2}} \frac{\sqrt{2}dx}{x(2-x^2)^{1/2}}.$$

Letting $x = \sqrt{2}\sin\theta$ so that $dx = \sqrt{2}\cos\theta d\theta$, we have

$$I = 2\int_0^{\pi/2} cosec\theta d\theta = -2\ln[cosec\theta + \cot\theta]_0^{\pi/2} \to \infty, \quad \text{at } \theta = 0.$$

Therefore it takes infinite length of time to go round the homoclinic orbit.

2 Non-Conservative Systems

We can also consider dissipative systems, eg.

1. The damped pendulum: $\ddot{x} + k\dot{x} + F(x) = 0$.
2. The Van der Pol oscillator: $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$, where $\mu \geq 0$.
3. The Lorenz equations:

$$\dot{x} = \sigma(y - x) = f$$

$$\dot{y} = rx - y - xz = g$$

$$\dot{z} = xy - bz = h,$$

where $\sigma, b, r > 0$ are real parameters. Here

$$\underline{\nabla}.\underline{f} = f_x + g_y + h_z = -\sigma - 1 - b < 0,$$

so volumes contract in phase space.

Such non-conservative systems have equilibria which are nodes and foci (stable and unstable), as well as isolated closed periodic trajectories called limit cycles. Other more complicated flows are possible such as quasi-periodic, chaotic and period-doubling solutions (see later).