LECTURE 2: NON-CONSERVATIVE SYSTEMS

We can also consider dissipative systems, eg.

1. The damped pendulum: $\ddot{x} + k\dot{x} + F(x) = 0$.
2. The Van der Pol oscillator: $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$, where $\mu \geq 0$.
3. The Lorenz equations:

$$\dot{x} = \sigma(y - x) = f$$

$$\dot{y} = rx - y - xz = g$$

$$\dot{z} = xy - bz = h,$$

where $\sigma, b, r > 0$ are real parameters. Here

$$\underline{\nabla}.\underline{f} = f_x + g_y + h_z = -\sigma - 1 - b < 0,$$

so volumes contract in phase space.

Such non-conservative systems have equilibria which are nodes and foci (stable and unstable), as well as isolated closed periodic trajectories called limit cycles. Other more complicated flows are possible such as quasi-periodic, chaotic and period-doubling solutions (see later).

2.1 Lyapunov Functions

We often wish to find conditions on the size of a region in which a fixed point is asymptotically stable, i.e. all trajectories approach it as $t \to \infty$. This leads to a generalisation of energy for conservative systems: the Lyapunov function.

Definition: Strong Lyapunov function

Consider the system $\dot{\underline{x}} = \underline{f}(\underline{x})$, with an equilibrium at $\underline{x} = \underline{x}_e$. If there exists a continuously differentiable function $V(\underline{x})$ of n variables x_j in some neighbourhood, D, of $\underline{x}_e$ such that

1. $V(\underline{x}) > 0$, i.e. positive definite $\forall \underline{x} \neq \underline{x}_e$ but with $V(\underline{x}_e) = 0$;
2. $\frac{dV}{dt} = \sum_j \frac{\partial V}{\partial x_j} f_j < 0$ is negative definite $\forall \underline{x} \neq \underline{x}_e$,

then $\underline{x}_e$ is globally asymptotically stable and the system cannot have closed orbits. Examples include a stable focus or a stable node.

2.2 Region of Asymptotic Stability

It is often possible to find a domain D in which V is a strong Lyapunov function in D. Then D is called the region of asymptotic stability for the equilibrium state.

Example: Consider the system:

$$\dot{x} = -x[1 - (x^2 + y^2)],$$

$$\dot{y} = -y[1 - (x^2 + y^2)],$$

for the equilibrium solution $(x, y) = (0, 0)$. Try $V(x, y) = x^2 + y^2$ for the Lyapunov function. Then $V > 0$ for all x, y except at $\underline{x}_e = \underline{0}$. Therefore V is positive definite and

$$\frac{dV}{dt} = V_x\dot{x} + V_y\dot{y}$$

$$= 2x[-x[1 - (x^2 + y^2)] + 2y[-y(1 - (x^2 + y^2))]]$$

$$= -2(x^2 + y^2)(1 - (x^2 + y^2)).$$

Thus $\dot{V} < 0$ for $x^2 + y^2 < 1$ and $\underline{x} \neq \underline{x}_e = \underline{0}$. Therefore $\dot{V}$ is negative definite inside the unit disc, D. Hence $\underline{x}_e = \underline{0}$ is globally asymptotically stable inside the region of asymptotic stability: the unit disc, D.

NB: If V is positive definite but $\dot{V} > 0$, then $\underline{x}_e$ is unstable.

Example: Show that $(0,0)$ is an unstable solution of

$$\dot{x} = x + xy^2$$

$$\dot{y} = y + x^2y.$$

Here $\dot{x} = 0 = \dot{y} \Rightarrow (0,0)$ is the only equilibrium state with

$$J_{\underline{0}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore $\lambda = 1$ (twice) and hence is unstable. Taking the Lyapunov function to be $V = x^2 + y^2$, V is positive definite. $\dot{V}$ is also positive definite since

$$\dot{V} = 2x\dot{x} + 2y\dot{y}$$

$$= 2x(x + xy^2) + 2y(y + x^2y)$$

$$= 2(x^2 + y^2) + 4x^2y^2,$$

and $(0,0)$ is unstable.

2.3 Poincaré-Bendixson Theorem

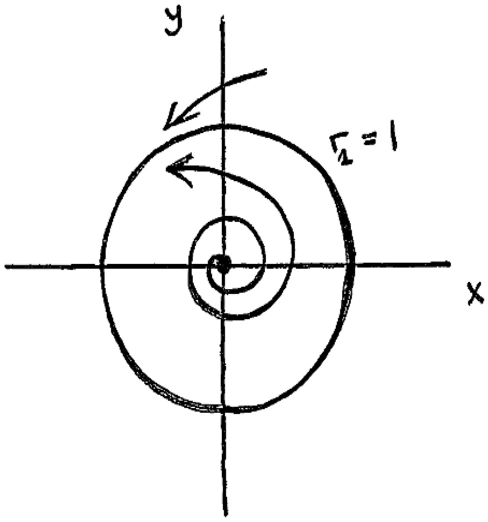
Suppose

1. $R \subset \mathbb{R}^2$ is a closed and bounded region.
2. $\underline{\dot{x}} = \underline{f}(\underline{x})$ is continuous and differentiable on an open set contained in R.
3. R does not contain any equilibria.
4. $\exists$ a trajectory C that is contained in R.

Then either C is a closed orbit, or it spirals towards one as $t \to \infty$. (See Jordan and Smith for a proof)
The Lyapunov function can also be used to show the existence of a limit cycle.

Example:
Consider $\dot{r} = r(1 - r^2) + \mu r\cos\theta$, $\dot{\theta} = 1$ where $r = (x^2 + y^2)^{1/2}$.

(A) For $\underline{\mu = 0}$: $\dot{r} = 0 \Rightarrow r_1 = 1, r_0 = 0$ are equilibrium solutions. $r_0 = 0 \Rightarrow x = y = 0$, ie. origin, while $r_1 = 1 \Rightarrow x^2 + y^2 = 1$, gives the unit circle.
In fact $r_1 = 1$ is a limit cycle. When $f(r) = r(1 - r^2)$, $f' = 1 - 3r^2 = \lambda$ (the eigenvalue, determining stability).



For $r_0 = 0$: $f' = \lambda_0 = 1 > 0$ and is therefore unstable.
For $r_1 = 1$, $f' = -2 < 0$ and is therefore stable.
We take Lyapunov function to be : $V = r^2$. Then $V(r)$ is positive definite in the neighbourhood of the origin. Also

$$\dot{V} = 2r\dot{r} = 2r^2(1 - r^2) \begin{cases} > 0, & \text{for } r^2 < 1 \\ < 0, & \text{for } r^2 > 1 \end{cases}.$$

Now for $r^2 < 1$, trajectories point outwards ($\dot{r} > 0$), while for $r^2 > 1$, trajectories point inwards ($\dot{r} > 0$). Therefore $r = 1$ corresponds to the limit cycle $\dot{r} = 0$.

(B) $\underline{\mu \neq 0}$, $1 >> \mu > 0$: We seek concentric circles of radii r_{min} and r_{max} such that $\dot{r} < 0$ on r_{max} (and the trajectory points inwards) and $\dot{r} > 0$ on r_{min} (and the trajectory points outwards). Then $0 < r_{min} \leq r \leq r_{max}$ defines a 'trapping region' R. Since $\dot{\theta} > 0$, there are no equilibria in R. Therefore the Poincaré - Bendixson Theorem guarantees a stable closed orbit if we can find r_{min} and r_{max}.

For $\underline{r_{min}}$:
We require $\dot{r} > 0$ i.e.

$$r(1 - r^2) + \mu r \cos\theta > 0, \quad \forall\theta.$$

Since $\cos\theta \geq -1$, we have

$$r(1 - r^2 + \mu\cos\theta) \geq r(1 - r^2 - \mu)) > 0$$

provided $r_{min} < \sqrt{1-\mu}$.

For $\underline{r}_{max}$:

$\dot{r} < 0 \Rightarrow (1 - r^2 - \mu < 0)$ ie.

$$r_{max} > \sqrt{1-\mu}.$$

Therefore we obtain the trapping region

$$r_{min} < \sqrt{1-\mu} < r_{max}.$$