LECTURE 5: NONLINEAR OSCILLATIONS

The equations we consider will be nonlinear perturbations of the equation for a simple harmonic oscillator.

There are a number of approaches for generating solutions of nonlinear oscillator equations which have a small parameter. Here we consider three such methods: the Poincaré- Lindstedt method; the Method of Multiple Scales; and the Krylov-Bogoliubov Method of Averaging, illustrating the procedures using the Duffing and the Van der Pol equations as examples. We begin with the the Poincaré-Lindstedt method (See e.g. Jordan and Smith).

1.1 Duffing's Equation

Consider Duffing's Equation for a nonlinear oscillator:

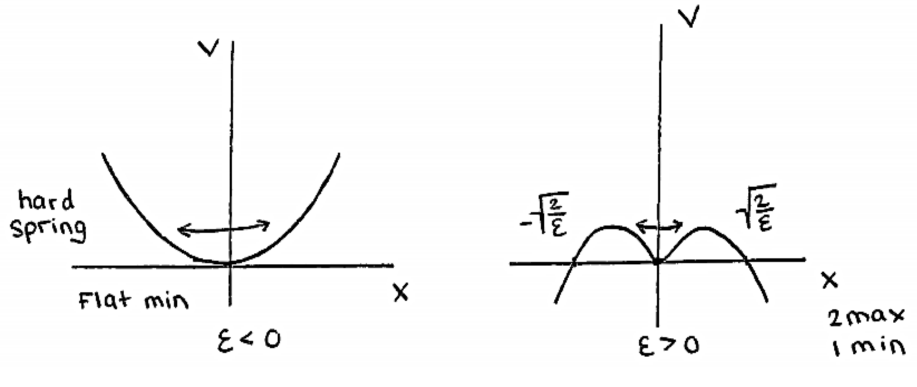
$$\ddot{x} + x - \epsilon x^3 = 0, \tag{1.1}$$

which governs oscillations of a spring with softening ($\epsilon > 0$) or hardening ($\epsilon < 0$). In what follows, we shall assume that $|\epsilon| \ll 1$. Multiplying through by $\dot{x}$ and integrating with respect to t we obtain the Conservation of Energy equation:

$$\frac{1}{2}\dot{x}^2 = V(x) = E,$$

where the potential function, $V(x)$ is

$$V = \frac{1}{2}x^2 - \frac{1}{4}\epsilon x^4 :$$



In either case, there exists periodic solutions. For small x (5.1) gives $\ddot{x} + x = 0$, so we expect an approximation to SHM i.e. $x \sim \cos t$, but with nonlinear corrections to the frequency $\omega = 1$. Therefore we set $\omega t = \tau$ and expand ω in powers of ϵ

$$\omega = 1 + \epsilon\omega_1 + \cdots . \tag{1.2}$$

Then (5.1), using $\frac{d}{dt} = \omega\frac{d}{d\tau}$, (5.1) becomes

$$\omega^2 x'' + x - \epsilon x^3 = 0. \tag{1.3}$$

We seek periodic solutions via a perturbation expansion in ϵ:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots, \tag{1.4}$$

and substitute (5.2) and (5.4) into (5.3) to obtain:

$$(1 + \epsilon\omega_1 + \cdots)^2(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)'' + x_0 + \epsilon x_1 + \cdots - \epsilon(x_0 + \epsilon x_1 + \cdots)^3 = 0,$$

or

$$(1 + 2\epsilon\omega_1 + \cdots)(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)'' + x_0 + \epsilon x_1 + \cdots - \epsilon(x_0^3 + 3x_0^2\epsilon x_1 + \cdots) = 0.$$

This generates a hierarchy of problems as coefficients of ϵ^r. Equating each to zero we have:
$\mathcal{O}(\epsilon^0)$: $x_0'' + x_0 = 0$
$\mathcal{O}(\epsilon^1)$: $x_1'' + x_1 = -2w_1 x_0 + x_0^3$, etc.

Choosing initial conditions such that $x_0(0) = a_0, x_0'(0) = 0$ and $x_n(0) = 0 = x_n'(0) = 0, \forall n > 0$, we obtain $x_0 = a_0 \cos\tau$ so that

$$\begin{aligned} x_1'' + x_1 &= -2\omega_1 a_0 \cos\tau + a_0^3 \cos^3\tau \\ &= -2\omega_1 a_0 \cos\tau + a_0^3\Big[\frac{3}{4}\cos\tau + \frac{1}{4}\cos\tau\Big] \\ &= \cos\tau\Big[-2\omega_1 a_0 + \frac{3}{4}a_0^3\Big] + \frac{a_0^3}{4}\cos^3\tau. \end{aligned}$$

Solving the homogeneous equation for x_1 gives solutions in $\cos\tau$ and $\sin\tau$, and a particular integral $\propto \tau\cos\tau, \tau\sin\tau$.
As τ increases, so do $\tau\cos\tau$ and $\tau\sin\tau$, rendering the expansion invalid (recall: we wrote $x_0 + \epsilon x_1$ and expected $|\epsilon| << x_0$ for all time). The coefficient of $\cos\tau$ on RHS of the $\mathcal{O}(\epsilon^1)$ is called a secular term and must be set to zero in order to maintain a uniformly valid perturbation expansion in ϵ. Thus we choose

$$\omega_1 = -\frac{3}{8}a_0^2,$$

which gives

$$x_1 = -\frac{1}{32}a_0^3\cos 3\tau$$

(we can always absorb $\cos\tau$ into x_0). We therefore obtain the solution:

$$x \sim a_0 \cos\tau - \frac{\epsilon}{32} a_0^2 \cos 3\tau + \cdots$$

with

$$\omega \sim 1 - \frac{3}{8}\epsilon a_0^2.$$

1.2 Van der Pol Oscillator

Now consider the equation for a Van der Pol oscillator:

$$\ddot{x} + x = \epsilon(1 - x^2)\dot{x}, \tag{1.5}$$

for $\epsilon << 1$. When $\epsilon = 0$, we again have the equation for SHM. Following the above, we write $\omega t = \tau$, and expand ω to get: $\omega = 1 + \epsilon\omega_1 + \cdots$, with $\frac{d}{dt} = \omega\frac{d}{d\tau}$.

Then (5.5) becomes

$$\omega^2 x'' + x = \epsilon(1 - x^2)\omega x'. \tag{1.6}$$

Looking for a solution:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots,$$

(5.6) becomes

$$(1 + \epsilon\omega_1 + \cdots)^2(x_0'' + \epsilon x_1'' + \cdots) + x_0 + \epsilon x_1 + \cdots$$

$$= \epsilon\Big(1 - (x_0 + \epsilon x_1 + \cdots)^2\Big)(1 + \epsilon\omega_1 + \cdots)(x_0' + \epsilon x_1' + \cdots),$$

so that

$$(1 + 2\epsilon\omega_1 + \cdots)(x_0'' + \epsilon x_1'' + \cdots) + x_0 + \epsilon x_1 + \cdots$$

$$= \epsilon\Big(1 - x_0^2 - 2\epsilon x_0 x_1 + \cdots\Big)(1 + \epsilon\omega_1 + \cdots)(x_0' + \epsilon x_1' + \cdots)(x_0' + \epsilon w_1 x_0' + \epsilon x_1).$$

As before we obtain a hierarchy of problems to be solved for the x_j.
$\mathcal{O}(1) : x_0'' + x_0 = 0,$
so that $x_0 = a_0 \cos\tau$, assuming initial conditions: $x_0(0) = a_0$, $x_j(0) = 0$ for all $j \neq 0$, $x_j'(0) = 0$ for all j.
At $\mathcal{O}(\epsilon)$ we have:

$$x_1'' + x_1 = -2\omega_1 x_0'' + x_0 - x_0^2 x_0'$$

$$= 2\omega_1 a_0 \cos\tau - a_0 \sin\tau + a_0^3 \cos^2\tau \sin\tau,$$

$$= 2\omega_1 a_0 \cos\tau - a_0\Big(1 - \frac{a_0^2}{4}\Big)\sin\tau + \frac{a_0^3}{4}\sin 3\tau,$$

using $\sin^3\tau = \frac{3}{4}\sin\tau - \frac{1}{4}\sin 3\tau$.

There are two types of secular terms, causing resonance here:
(i) terms $\propto \cos\tau$, which requires us to set $\omega_1 = 0$.
(ii) terms $\propto \sin\tau$ which requires us to set $a_0 = 2$.
Therefore we obtain the periodic solution $x = 2\cos\tau$, with amplitude '2', and frequency $\omega \sim 1 + \mathcal{O}(\epsilon^2)$.

1.3 Method of Multiple Scales

Instead of working with one time scale, in the Method of Multiple Scales we work with two. For Duffing's equation we saw the need to remove secular terms $\propto \tau\cos\tau$ and $\tau\sin\tau$ to avoid the expansion

$$x \approx x_0 + \epsilon x_1 + \cdots$$

becoming invalid when $\epsilon x_1 = \mathcal{O}(x_0)$. This would happen as τ increases.

We now introduce a new timescale $T = \epsilon t$ and take $x = x(t, T)$, where t is a 'fast' time scale and T is a 'slow' time scale, chosen to remove the secular terms. Then

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial T}, \tag{1.7}$$

and we seek a solution in which the amplitude of each term now depends upon both time scales t, T:

$$x = x_0(t,T) + \epsilon x_1(t,T) + \epsilon^2 x_2(t,T) + \cdots. \tag{1.8}$$

1.4 Duffing's equation revisited

Consider Duffing's equation in the form

$$\ddot{x} + x + \epsilon x^3 = 0. \tag{1.9}$$

Then, using (6.1), Duffing's equation becomes

$$x_{tt} + 2\epsilon x_{tT} + \epsilon^2 x_{TT} + x + \epsilon x^3 = 0. \tag{1.10}$$

Substituting the expansion (6.2) for x, we obtain:

$$x_{0tt} + \epsilon x_{1tt} + \cdots + 2\epsilon x_{0tT} + \cdots + x_0 + \epsilon x_1 + \epsilon x_0^3 + \cdots = 0. \tag{1.11}$$

As before we equate corresponding powers of ϵ to zero. It is often easier to work with complex amplitudes. Here is a case in point. At
$\mathcal{O}(\epsilon^0)$: $x_{0tt} + x_0 = 0$ which suggests we write

$$x_0 = A(T)e^{it} + A^*(T)e^{-it},$$

where '*' denotes taking the complex conjugate. The amplitude $A(T)$ now depends upon the slow time scale T (c.f. the Poincaré-Lindstedt method where the amplitude was a constant). Proceeding to the $\mathcal{O}(\epsilon)$ problem we have:

$$x_{1tt} + x_1 = -2x_{0tT} - x_0^3$$

$$= -2(iA_T e^{it}) - (A^3e^{3it} + 3A^2A^* + cc)$$

$$= (2iA_T + 3|A|^2A)e^{it} - A^3e^{3it} + cc.$$

The left hand side has a homogeneous solution $\propto e^{it}$. It therefore follows that the particular solution is $\propto te^{it}$, and hence is a secular term, which must be removed to render the expansion uniformly valid. This gives

$$-3|A|^2A = 2iA_T \qquad \Rightarrow A_T = \frac{3}{2}i|A|^2A. \tag{1.12}$$

We seek a separable solution to (5.12) of the form $A(T) = r(T)e^{i\theta(T)}$. Then, upon equating real and inmaginary parts, (5.12) becomes

$$r_T = 0, \tag{1.13a}$$
$$\theta_T = \frac{3}{2}r^2. \tag{1.13b}$$

Hence $r(T) = r_0$ and $\theta(T) = \frac{3}{2}{r_0}^2T + \theta_0$ Thus

$$A(T) = r_0 e^{(i\frac{3}{2}{r_0}^2T+\theta_0)}$$

so that

$$x_0 = r_0 e^{(i\frac{3}{2}{r_0}^2T+\theta_0+t)} + c.c.$$

or

$$x_0 = 2r_0 cos(\frac{3}{2}{r_0}^2T + \theta_0 + t)$$

.

Given the initial condition, $x(0) = a_0$, it follows that $2r_0 = a_0$ and taking $\theta_0 = 0$, we obtain

$$x_0 = cos(1 + \frac{3}{8}\epsilon {a_0}^2)t + \mathcal{O}(\epsilon^2).$$

as before.

1.5 Van der Pol via Multiple Scales

Consider the Van der Pol equation (5.5):

$$\ddot{x} + x + \epsilon(x^2 - 1)\dot{x} = 0, \quad \epsilon << 1.$$

As before, we introduce the slow time scale $T = \epsilon t$ such that $\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial T}$ and take $x = x(t, T)$. Then the Van der Pol equation becomes

$$x_{tt} + 2\epsilon x_{tT} + \epsilon^2 x_{TT} + x + \epsilon(x^2 - 1)(x_t + \epsilon x_T) = 0. \tag{1.14}$$

Again we seek a solution of the form:

$$x = x_0(t, T) + \epsilon x_1(t, T) + \epsilon^2 x_2 + \cdots,$$

and substitute into (5.13) to get

$$x_{0tt}+\epsilon x_{1tt}+\cdots+2\epsilon x_{0tT}+\cdots+x_0+\epsilon x_1+\cdots+\epsilon(x_0^2+2\epsilon x_0 x_1+\cdots-1)(x_{0t}+\epsilon x_{1t}+\epsilon x_{0T}) = 0. \tag{1.15}$$

As before

$\mathcal{O}(1)$: $x_{0tt} + x_0 = 0$
$\mathcal{O}(\epsilon)$: $x_{1tt} + x_1 = -2x_{0tT} - (x_0^2 - 1)x_{0t}$, etc.

For the $\mathcal{O}(1)$ problem we take the solution: $x_0 = A(T)e^{it} + A^*(T)e^{-it}$.
Therefore

$$\begin{aligned} x_{1tt} + x_1 &= -2A_T ie^{it} + c.c. - (A^2e^{2it} + 2|A|^2 + A^{*2}e^{-2it} - 1)(iAe^{it} - iA^*e^{-it}) \\ &= -e^{it}i\Big(2A_T + A|A|^2 - A\Big) - ie^{3it}\Big(A^3\Big) + c.c. \end{aligned}$$

Again secular terms exist on the RHS, which would generate a solution $\propto te^{it}$, leading to a breakdown of the regular perturbation expansion, unless we choose the coefficient of e^{it} to be zero, namely

$$A_T = \frac{1}{2}A(1 - |A|^2), \tag{1.16}$$

which gives an evolution equation for $A(T)$.

If we again write $A(T) = r(T)e^{i\theta(T)}$ and substitute into (5.16), we can equate real and imaginary parts to get

$$r_T = \frac{1}{2}r(1 - r^2), \tag{1.17a}$$
$$\theta_T = 0. \tag{1.17b}$$

Thus $\theta(T) = \theta_0$ and $r(T) = (1 + Cexp(-T))^{-1/2}$, so that

$$x_0 = r(e^{i(\theta_0 + t)} + c.c.)$$

or

$$x_0 = 2rcos(\theta_0 + t) \quad \rightarrow \quad 2cos(\theta_0 + t)$$

as $t \rightarrow \infty$.

We again obtain the limit cycle with amplitude 2.

The third method we consider is