LECTURE 8:

SYNCHRONISATION OF COUPLED OSCILLATORS

Consider 2-D phase space on a torus: $S' \times S'$ for systems of the form

$$\dot{\theta}_1 = f_1(\theta_1, \theta_2), \quad (4.1a)$$

$$\dot{\theta}_2 = f_2(\theta_1, \theta_2), \quad (4.1b)$$

where f_j are periodic in θ_j.

Such systems are called 'phase oscillators', where θ_j are phase variables. For example consider the coupled phase oscillator system:

$$\dot{\theta}_1 = \omega_1 + k_1 \sin(\theta_2 - \theta_1), \quad (4.2a)$$

$$\dot{\theta}_2 = \omega_2 + k_2 \sin(\theta_1 - \theta_2). \quad (4.2b)$$

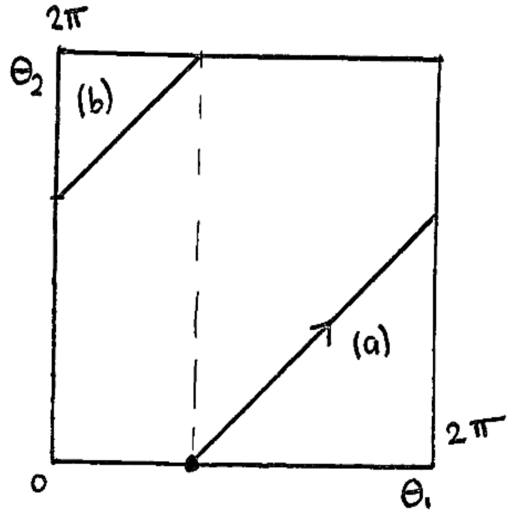
Here $w_j > 0$ are the linear natural frequencies and $k_j \geq 0$ are coupling constants. We consider two cases.

1. $\underline{k_j = 0}$, Here (8.2) decouples and trajectories are solutions to

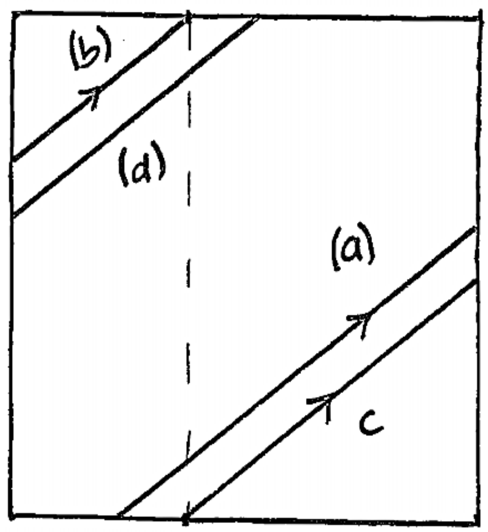
$$\frac{\dot{\theta}_2}{\dot{\theta}_1} = \frac{\omega_2}{\omega_1} \quad \Rightarrow \frac{d\theta_2}{d\theta_1} = \frac{\omega_2}{\omega_1},$$

i.e.

$$\theta_2 = \frac{\omega_2}{\omega_1}\theta_1 + c$$



If $\frac{\omega_1}{\omega_2} = \frac{p}{q}$ is rational, we have a 'frequency locked periodic state' on a two-torus: a periodic limit cycle.



However if $\frac{\omega_1}{\omega_2}$ is irrational, orbits fill the whole torus, and we have a 'quasi-periodic' solution.

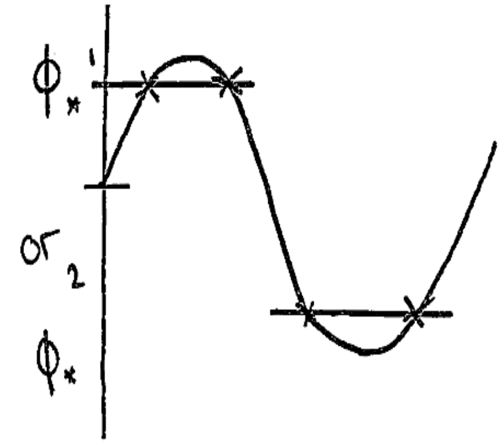
2. $\underline{k_j \neq 0}$: Let $\phi = \theta_1 - \theta_2$. Then it follows from (8.2) that

$$\dot{\phi}(= \dot{\theta}_1 - \dot{\theta}_2) = (\omega_1 - \omega_2) - (k_1 + k_2)\sin\phi. \tag{4.3}$$

When $\dot{\phi} = 0$, we have

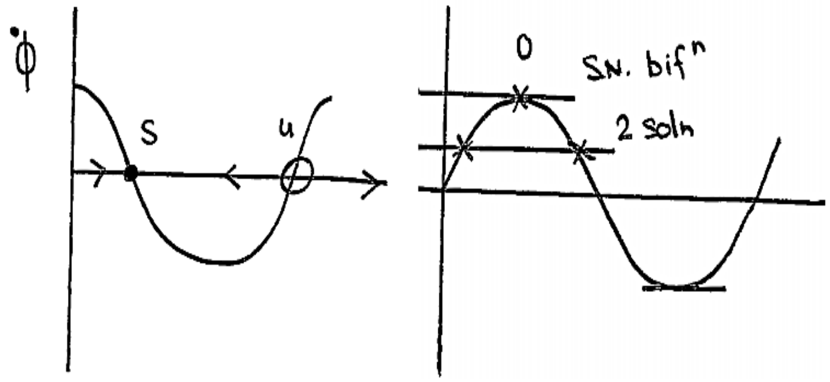
$$\sin\phi_* = \frac{(\omega_1 - \omega_2)}{k_1 + k_2}, \tag{4.4}$$

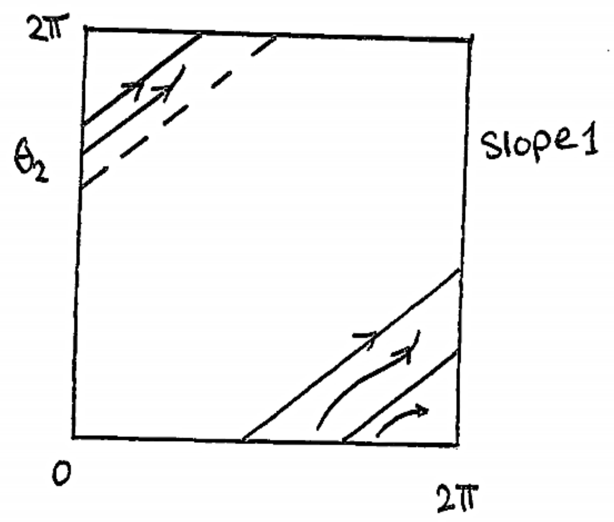
from which follows that there are two equilibrium states for $|\omega_1 - \omega_2| \leq k_1 + k_2$ and zero equilibria for $|\omega_1 - \omega_2| > k_1 + k_2$.



NB: A saddle-node bifurcation of limit cycles occurs when $\sin\phi_* = 1$.

The equilibrium state given by (8.4) corresponds to a phase-locked solution.





When phase-locked, both oscillators run at the same frequency:

$$\dot{\theta}_1 = \dot{\theta}_2 = w_j + k_j \sin\phi^* = w_2 + \frac{(w_1 - w_2)}{k_1 + k_2} k_2 = \frac{k_1 w_1 + k_2 w_2}{k_1 + k_2}. \qquad (4.5)$$

Example 1: N-Dimensional damped Josephson Junctions.
Consider

$$\dot{\phi}_i(t) = \Omega + a \sin\phi_i + \frac{1}{N} \sum_{j=1}^{N} \sin\phi_j.$$

When $\phi_1 = \phi_2 = \cdots = \phi_N = \phi^*(t)$, we have an in-phase solution; and it follows that

$$\dot{\phi}^* = \Omega + a\sin\phi^* + \sin\phi^*.$$

As above, there exist periodic solutions $\iff$ $|\Omega| < |a+1|$.

To study stability, we perturb $\phi_i = \phi^* + \eta_i$ to get

$$\dot{\eta}_i = (a\cos\phi^*)\eta_i + \cos(\phi^*)\frac{1}{N}\sum_{j=1}^{N} \eta_j.$$

Introducing $\xi_i = \eta_{i+1} - \eta_i$, it follows that

$$\dot{\xi}_i = \dot{\eta}_{i+1} - \dot{\eta}_i = (a\cos\phi^*)(\eta_{i+1} - \eta_i) = (a\cos\phi^*)\xi_i.$$

Therefore

$$\frac{d\xi_i}{\xi_i} = (a\cos\phi^*)dt = \frac{(a\cos\phi^*)d\phi^*}{\Omega + (a+1)\sin\phi^*}$$

and hence

$$\oint \frac{d\xi_i}{\xi_i} = \int_0^{2\pi} \frac{(a\cos\phi^*)d\phi^*}{\Omega + (a+1)\sin\phi^*},$$

so that

$$\ln\left(\frac{\xi_i(T)}{\xi_i(0)}\right) = \frac{a}{1+a}\ln\left[\Omega + (a+1)\sin\phi^*\right]_0^{2\pi} = 0.$$

Thus $\xi_i(T) = \xi_i(0)\forall i$, and we have neutral stability.

Example 2: Consider N phase oscillators with nearest neighbour coupling:

$$\begin{aligned}
\dot{\theta}_1 &= \omega_1 + a\sin(\theta_2 - \theta_1), && (4.6a)\\
\dot{\theta}_i &= \omega_i + a[\sin(\theta_{i+1} - \theta_i) + \sin(\theta_{i-1} - \theta_i)], \quad (i = 2, \cdots N-1) && (4.6b)\\
\dot{\theta}_N &= \omega_N + a\sin(\theta_{N-1} - \theta_N). && (4.6c)
\end{aligned}$$

Let $\phi_i = \theta_i - \theta_{i+1}$, $i = 1, \ldots, N-1$. Then we get

$$\underline{\dot{\phi}} = \underline{\Omega} + A\underline{S},$$

where

$$\underline{\phi} = [\phi_1, \ldots, \phi_{N-1}]^T,$$

$$\underline{\Omega} = [\omega_1 - \omega_2, \ldots, \omega_{N-1} - \omega_N]^T,$$

$$\underline{S} = [\sin\phi_1, \ldots, \sin\phi_{N-1}]^T,$$

and A is the tri-diagonal matrix:

$$A = \alpha\begin{bmatrix} -2 & 1 & 0 & \cdots & \cdots & 0\\ 1 & -2 & 1 & \cdots & \cdots & 0\\ 0 & 1 & -2 & 1 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 0 & \cdots & \cdots & 0 & 1 & -2 \end{bmatrix}.$$

Since $A_{ij} = A_{ji}$, A^{-1} exists and is symmetric.

N.B.:

(a) The equilibria are all phase-locked orbits with $\dot{\phi}_i = 0, \forall i$.

(b) $\underline{\dot{\phi}} = 0 \Rightarrow \underline{S} = -A^{-1}\underline{\Omega}$ provided each component of $A^{-1}\underline{\Omega}$ is ≤ 1.

(c) $A^{-1}_{ij} = \frac{j(N-1)}{-N\alpha}$, $(i \geq j)$.

(d) If $\omega_1 = \omega$, $\omega_2 = \omega - \Delta$, $\omega_3 = \omega - 2\Delta$, etc, then it follows that $\underline{\Omega} = \Delta[1, \cdots, 1]^T$.

(e) At equilibrium: $\sin\phi_i = \Delta i(N-i)/2\alpha$, for $i = 1, \ldots, N-1$. Therefore

$$|\sin\phi_i| \leq 1 \quad \Rightarrow \left|\frac{\Delta i(N-i)}{2\alpha}\right| \leq 1, \quad \text{i.e.} \quad \left|\frac{\Delta}{\alpha}\right| \leq \frac{2}{i(N-i)}.$$

For example, when $N = 6$, we have

$$A^{-1} = -\frac{1}{6\alpha}\begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}.$$

Therefore

$$\underline{S} = \begin{bmatrix} \sin\phi_1 \\ \vdots \\ \vdots \\ \sin\phi_5 \end{bmatrix} = \frac{\Delta}{2\alpha}\begin{bmatrix} 5 \\ 8 \\ 9 \\ 8 \\ 5 \end{bmatrix} \quad \Rightarrow \left|\frac{\Delta}{2\alpha}\right| \leq \frac{1}{9}$$

Synchronisation

Phase locking is an example of synchronisation. Kuramoto showed that any system of weakly coupled, nearly identical limit cycle oscillators had long time dynamics, governed by equations of the form

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i), \quad i = 1, \quad , N.$$

NB: Here the functions Γ_{ij} are determined from integrals. The simplest case has

$$\Gamma_{ij}(\theta_j - \theta_i) = \frac{k}{N}\sin(\theta_j - \theta_i)$$

for $k \geq 0$.