LECTURE 9:

POINCARÉ MAPS

Another way of representing phase space is by looking at cross-sections of trajectories and is particularly useful for non-autonomous systems. It reduces the dimension of a system by one and replaces the system of ODEs by discrete difference equations or maps.

Consider the autonomous system

$$\dot{\underline{x}} = \underline{f}(\underline{x})$$

with phase portrait:

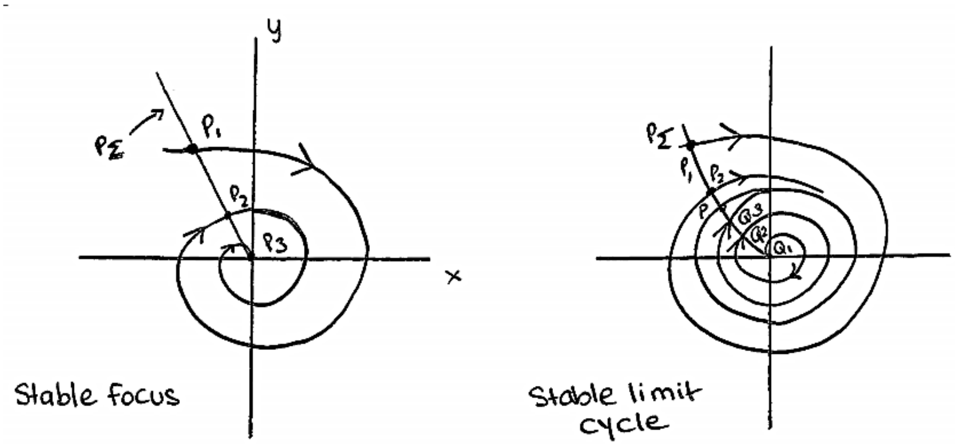


Figure 1: (A) stable focus and (B) stable limit cycle

Definition 1. *A Poincaré section, P_Σ, is a transverse section of the trajectories, which is nowhere tangential to any trajectory.*

Here the trajectories cut P_Σ at points: $P_1, P_2, \ldots, P_n$ and $Q_1, Q_2, \ldots, Q_n$.

Definition 2. *The Poincaré Map (or 1st-return Map)*

$$P : P_\Sigma \to P_\Sigma$$

is a map which sends P_i on P_Σ to P_{i+1}, or Q_i to Q_Σ on the Poincare section P_Σ.

N.B.

1. $(x_2, y_2) = \mathcal{P}(x_1, y_1)$; $(x_3, y_3) = \mathcal{P}(x_2, y_2) = \mathcal{P}^2(x_1, y_1)$. Therefore we have that $(x_n, y_n) = \mathcal{P}^n(x_1, y_1)$, where $\mathcal{P}^n$ is the n^{th} iterate of $\mathcal{P}$.

2. The stability of an equilibrium is determined from its 1st return map.

In the above Examples

(A) $\lim_{n\to\infty} \mathcal{P}^n(x_1, y_1) = (0,0)$ i.e. a stable focus.

(B) $\lim_{n\to\infty} \mathcal{P}^n(x_{p_1}, y_{p_1}) = (x_p, y_p) = \lim_{n\to\infty} \mathcal{P}^n(x_{q_1}, y_{q_1})$, where $\lim_{n\to\infty} \mathcal{P}^n(x_{p_1}, y_{p_1})$ is from the outside and $\lim_{n\to\infty} \mathcal{P}^n(x_{q_1}, y_{q_1})$ is from inside. Thus $P = (x_p, y_p)$ is a fixed point of the map and so corresponds to a closed periodic orbit (i.e. a limit cycle) of the original flow.

Example Find the first return map for

$$\dot{x} = \mu x - \omega y - x(x^2 + y^2)^{1/2}$$

$$\dot{y} = \omega x + \mu y - y(x^2 + y^2)^{1/2} \tag{1.1}$$

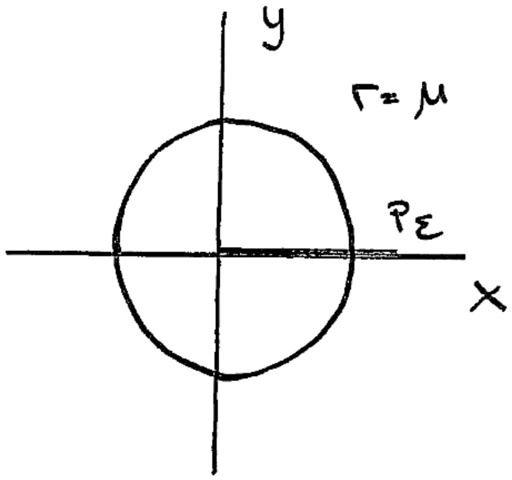
for a Poincare section given by $y = 0$, $x > 0$, $\mu > 0$.

Solution

In polar coordinate (9.1) becomes

$$\dot{r} = \mu r - r^2 = f(r), \tag{1.2a}$$

$$\dot{\theta} = \omega. \tag{1.2b}$$



(9.2)b $\Rightarrow \theta = \omega t + \theta o$,

while (9.2)a $\Rightarrow$ there exists two equilibria: $r = 0$ (ie. $(x, y) = (0,0)$) and $r = \mu$ (i.e. $x^2 + y^2 = \mu^2$).

Stability is determined from (9.2)a: $f' = \mu - 2r$, so that $\lambda = \mu - 2r$ For $r = 0$, $\lambda = \mu > 0$, i.e. unstable, whereas for $r = \mu$, $\lambda = -\mu < 0$ i.e. stable.

Trajectories are given by $r(\theta)$:

$$\frac{\dot{r}}{\dot{\theta}} = \frac{dr}{d\theta} = \frac{r}{\omega}(\mu - r) \Rightarrow \quad \int_{r_0} \frac{dr}{r(\mu - r)} = \int_{\theta_0}^{\theta} \frac{d\theta}{\omega}.$$

Therefore

$$r = \frac{\mu r_0}{[r_0 + (\mu - r_0)\exp -\frac{\mu}{\omega}(\theta - \theta_0)]}.$$

Now choose $\theta_0 = 0$ such that we start on $y = 0$, $x > 0$. Then $\theta = 0$. Each successive return map is a multiple of 2π. Since $\dot{\theta} > 0$, we go anticlockwise and each P_j is on $\theta_j = 2j\pi$, $j = 1, 2, \cdots$ with $P_1 = (r_0, 0)$. Thus the n^{th} return map is

$$r_n = \frac{\mu r_0}{r_0 + (\mu - r_0)e^{-2n\frac{\pi\mu}{\omega}}}$$

and the first return map is

$$r_1 = \frac{\mu r_0}{r_0 + (\mu - r_0)e^{-2\frac{\pi\mu}{\omega}}}.$$

1.1 1-D Iterative Maps

Consider iterations of a 1-D map

$$x_{n+1} = f(x_n; \mu), \tag{1.3}$$

where μ is a real parameter, f is a smooth nonlinear function, whose range $\subset$ domain. The equivalent of each equilibria for maps are fixed points.

Definition 3. *x_* is a fixed point (F.P.) of (9.3) if*

$$x_* = f(x_*; \mu),$$

i.e. $x_n = x_{n+1}$ for all n.

Stability: Writing $x = x_* + \xi$ we get

$$(x_{n+1} =) \quad x_* + \xi_{n+1} = f(x_* + \xi_n) \quad (= f(x_n)$$

$$= f(x_*) + \xi_n f'(x_*) + \mathcal{O}(\xi_n^2).$$

Therefore

$$x_{n+1} = f'(x_*)\xi_n. \tag{1.4}$$

Definition 4. *The multiplier of a map $x_{n+1} = f(x_n)$ with fixed point, x_*, is defined as $\lambda = f'(x_*)$ and determines its stability.*

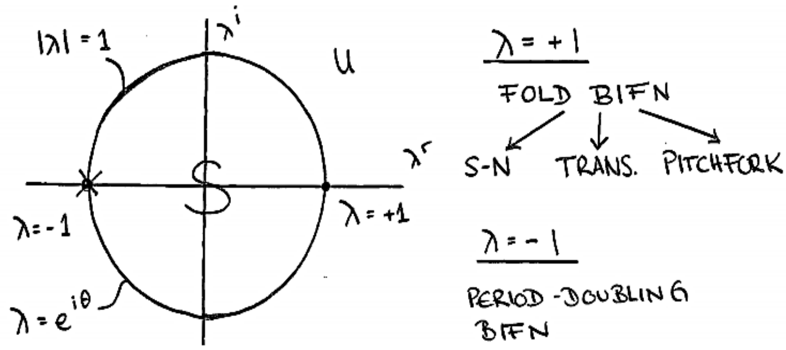
Iterating

$$x_1 = \lambda x_0 \Rightarrow \quad x_2 = \lambda x_1 = \lambda^2 x_0, \ldots, x_n = \lambda^n x_0.$$

If $|\lambda| < 1,$ the FP is stable since $\xi_n \to 0$ as $n \to 0.$

If $|\lambda| > 1,$ The FP is unstable.

If $|\lambda| = 1,$ we have neutral stability and bifurcations occur.



1.2 Linear stability of limit cycles

We can repeat the stability analysis of fixed points for limit cycles. Consider a limit cycle as a closed orbit of an associated Poincaré map $\mathcal{P}$:

$$\underline{x}^* = \mathcal{P}\underline{x}^*.$$

Let $\underline{v}_0$ be a small perturbation, such that $\underline{x}^* + \underline{v}_0 \in \mathcal{P}_\Sigma$.
Then after one return,

$$\underline{x}^* + \underline{v}_1 = \mathcal{P}(\underline{x}^* + \underline{v}_0).$$

Taylor-expanding we obtain

$$\underline{v}_1 = [D\mathcal{P}(\underline{x}^*)]\underline{v}_0,$$

where $D\mathcal{P}$ is the $(n-1)x(n-1)$ Jacobian matrix, called the linearised Poincaré map, evaluated at $\underline{x}^*$.
The eigenvalues of $D\mathcal{P}$ determine the stability of the periodic orbit. If $|\lambda_j| < 1 \quad \forall j = 1, ..., n-1$, the closed orbit is linearly stable.
N.B. (1) For maps, $|\lambda_j| < 1$ for stability (i.e. the λ_j all lie within the unit circle), c.f. ODEs, where we require $\Re(\lambda_j) < 0$ for stability.
(2) The λ_j are called characteristic multipliers or Floquet exponents of the periodic orbit.

(3) There is always an additional multiplier with $\lambda = 1$, corresponding to time translations along the periodic orbit.

1.3 Periodic Cycles

As well as FPs, maps have periodic cycles. Consider

$$x_{n+1} = f(x_n)$$

$$x_{n+2} = f(x_{n+1}) = f^2(x_n), \dots$$

$$x_{n+k} = f^k(x_n) \quad \text{kth iterate of the map.}$$

Definition 5. *$x = x_p$ is a periodic cycle of period k if*

$$f^k(x_p) = x_p \quad \text{but} \quad f^r(x_p) \neq x_p \quad \text{for} \quad r = 1, 2, \dots, k.$$

NB:

(i) Each point on the cycle satisfies $f^k(x_p) = x_p$.

(ii) For a period-2 cycle, $x_2 = f(x_1)$, $x_1 = f(x_2)$, and therefore

$$x_j = f^2(x_j), \quad j = 1, 2.$$

(iii) A FP is a period-1 cycle.

(iv) Chaos arises in nonlinear 1-D maps via cascades of period-doubling bifurcations.

Lemma 1. *All points of a k-cycle are fixed points of $f^k(x)$ with multiplier $\lambda_k = f'(x_1)f'(x_2)\dots f'(x_k)$.*

Proof. Suppose $x_{j+1} = f(x_j)$ is a point on the k-cycle. Then

$$f^k(x_{j+1}) = f^{k+1}(x_j) = f(f^k(x_j)) = f(x_j) = x_{j+1}.$$

Without loss in generality, we can therefore take x_1. By the chain rule:

$$\begin{aligned}
\lambda_k &= \frac{d}{dx} f^k(x_1) = \frac{d}{dx} f(f^{k-1}(x_1)) \\
&= f'(f^{k-1}(x_1)) \frac{d}{dx} f^{k-1}(x_1) \\
&= f'(x_k) \frac{d}{dx} f(f^{k-2}(x_1)) \\
&= f'(x_k) f'(f^{k-2}(x_1)) \frac{d}{dx} f^{k-2}(x_1) \\
&= f'(x_k) f'(x_{k-1}) \frac{d}{dx} f^{k-2}(x_1) \\
&= \dots \\
&= f'(x_k) f'(x_{k-1}) \dots \frac{d}{dx} f(f(x_1)) \\
&= f'(x_k) f'(x_{k-1}) \dots f'(f(x_1)) \frac{d}{dx} f(x_1) \\
&= f'(x_k) f'(x_{k-1}) \dots f'(x_2) f'(x_1). \qquad (1.5)
\end{aligned}$$

In particular for a two-cycle $k = 2$, since $x_2 = f(x_1)$ and $x_1 = f(x_2)$, $x_1 = f(f(x_1))$ and so, by the chain rule,

$$\lambda_2 = \frac{d}{dx} f(f(x_1)) = f'(f(x_1)) f'(x_1) = f'(x_2) f'(x_1). \qquad (1.6)$$