LECTURE 10:

BIFURCATIONS IN 1-D MAPS

Recall: In 1-D maps, bifurcations occur when $\lambda = 1$, $\lambda = -1$, or $\lambda = e^{i\theta}$. Here we focus on $\lambda = 1$ and $\lambda = -1$.

When $\lambda = +1$, we have a Fold bifurcation, which can be either a Saddle-Node, Transcritical or Pitchfork bifurcation. (In contrast to similar bifurcations in ODES, which occur when $\lambda = 0$).

When $\lambda = -1$, we have a Flip or Period-Doubling bifurcation.

We illustrate the procedure, via examples.

2.1 I: Saddle-Node

Consider

$$x_{n+1} = \mu + x_n - x_n^2 = f(x_n, \mu)$$

(i) Fixed points satisfy $x_{n+1} = x_n \quad \forall n$, so that:

$$x = f(x,\mu) \quad \Rightarrow \quad x = \mu + x - x^2.$$

Therefore $\mu = x^2$
For

$\mu < 0$ i.e. there are no fixed points,

$\mu = 0$ there are 2 fixed points, $x = 0$ (twice),

$\mu > 0$ there are 2 distinct fixed points, $x_\pm = \pm\sqrt{\mu}$.

We therefore have a saddle-node bifurcation.

(ii) The multiplier is:

$$\lambda = f'(x_\pm) = 1 - 2x_{x=x_\pm}, \quad \text{so that} \quad \lambda_\pm = 1 \mp 2\sqrt{\mu}.$$

NB. At $\mu = 0$, $\lambda_\pm = 1$ so we have a fold bifurcation.

(iii) Stability: For

$$x_+ = \sqrt{\mu} : \lambda_+ = 1 - 2\sqrt{\mu} < 1 \Rightarrow \text{ stable for small } \mu$$

$$x_- = -\sqrt{\mu} : \lambda_- = 1 + 2\sqrt{\mu} > 1 \Rightarrow \text{unstable.}$$

(iv) Range of stability: NB. $x_+ = \sqrt{\mu}$ is stable when the multiplier falls within the unit circle, i.e. for

$$-1 < \lambda_+ < 1$$

$$-1 < 1 - 2\sqrt{\mu} < 1$$

$$-2 < -2\sqrt{\mu} < 0$$

$$1 > \sqrt{\mu} > 0.$$

Therefore x_+ undergoes a fold (specifically a saddle-node) bifurcation at $\mu = 0$ and a period-doubling bifurcation at $\mu = 1$.

(v) Bifurcation Diagram

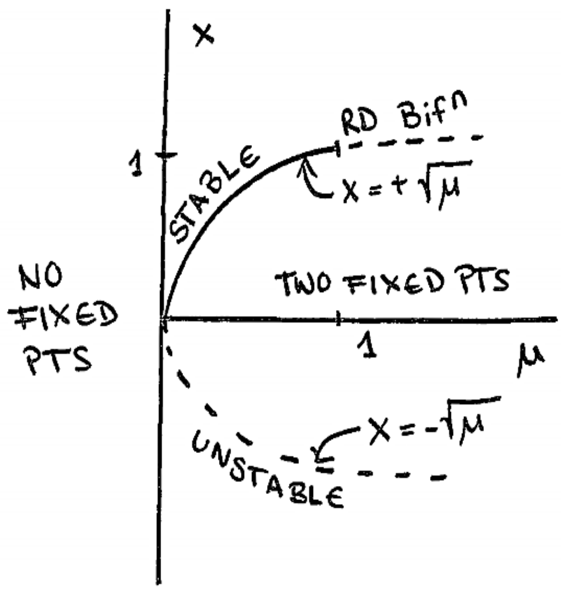


Figure 2: Saddle-node bifurcation diagram

2.2 II: Pitchfork Bifurcation

Consider

$$x_{n+1} = x_n(1+\mu) - x_n^3 = f(x_n, \mu).$$

Fixed points satisfy:

$$x_n = x_{n+1} = x : \quad x = x(1+\mu) - x^3,$$

so that $x = 0$ or $x^2 = \mu$.

$$\text{For } \mu < 0 : \quad \exists \quad 1 \text{ FP:} \quad x = 0$$

$$\mu = 0 : \quad \exists \quad 3 \text{ FP:} \quad x = 0 \quad (3 \text{ times})$$

$$\mu > 0 : \quad \exists \quad 3 \text{ distinct FP:} \quad x = 0, \quad x_{\pm} = \pm\sqrt{\mu}.$$

Multiplier:

$$\lambda = f'(x) = 1 + \mu - 3x^2.$$

We consider each fixed point in turn.

$\underline{x = 0}$: $\lambda_0 = 1 + \mu$ ($= 0$ for $\mu = 0 \Rightarrow$ we have a fold bifurcation). For

$$\mu > 0 \Rightarrow \quad \lambda_0 > 1 \Rightarrow x = 0 \text{ is unstable}$$

$$\mu < 0 \Rightarrow \quad \lambda_0 < 1 \Rightarrow x = 0 \text{ is stable for small } \mu$$

More specifically, the region of stability is given by :

$$-1 < 1 + \mu = \lambda_0 < 1, \quad \text{i.e.} \quad -2 < \mu < 0.$$

Therefore $x = 0$ undergoes a period-doubling bifurcation at $\mu = -2$ and a fold bifurcation at $\mu = 0$.

$\underline{x_\pm = \pm\sqrt{\mu}}$: Here $\lambda_\pm = 1 - 2\mu$ ie. both nontrivial fixed points have the same stability.

$$\mu > 0, \quad \lambda_\pm < 1 \Rightarrow \quad \text{stable}$$

$$\mu < 0, \quad x_\pm \quad \text{doesn't exist}$$

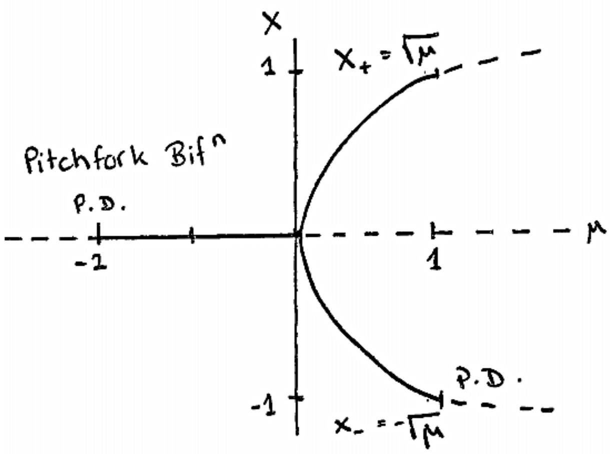
$$\mu = 0, \quad \lambda_\pm = 1 \quad \text{ie. fold bifurcation.}$$

$x_\pm$ are stable in

$$-1 < 1 - 2\mu < 1, \quad \text{i.e.} \quad 1 > \mu > 0.$$

So we have a fold (pitchfork) bifurcation at $\mu = 0$ and a period-doubling bifurcation at $\mu = 1$.

The bifurcation picture so far:



There are 3 period-doubling bifurcations since $\lambda \to -1$ for each FP.

We now investigate the p.d. bifurcation off $x = 0$. There are two methods.

Method 1: The p.d. bifurcation creates a P2-cycle, which is a fixed point of the second iterate of the map:

$$x_{n+2} = f^2(x_n, \mu) = x_n.$$

∴

$$x_{n+2} = (1+\mu)x_{n+1} - x_{n+1}^3$$

$$= (1+\mu)[(1+\mu)x_n - x_n^3] - [(1+\mu)x_n - x_n^3]^3$$

$$= x_n.$$

This gives a polynomial of degree 9 in x_n, 3 of whose roots are the fixed points of f ie. $x = 0$, $x = \pm\sqrt{\mu}$. It is hard to factor the remaining 6th degree polynomial for the rest.
Method 2: Find expressions for $x_1 + x_2$ and $x_1 x_2$.
Let x_1 and x_2 be the two points of the P2 cycle. Then $x_1 \neq x_2$ satisfy:

$$x_2 = (1+\mu)x_1 - x_1^3, \quad (2.1a)$$
$$x_1 = (1+\mu)x_2 - x_2^3. \quad (2.1b)$$

(10.1)a -(10.1)b ⇒

$$x_2 - x_1 = (1+\mu)(x_1 - x_2) - (x_1^3 - x_2^3),$$

and since $x_1 \neq x_2$,

$$-1 = (1+\mu) - (x_1^2 + x_1 x_2 + x_2^2).$$

Therefore

$$x_1^2 + x_1 x_2 + x_2^2 = 2 + \mu. \quad (2.2)$$

Also from (10.1)a × (10.1)b ⇒

$$1 = (1+\mu)^2 - (1+\mu)(x_1^2 + x_2^2) + x_1^2 x_2^2. \quad (2.3)$$

Substitute (10.2) into (10.3) gives:

$$x_1^2 x_2^2 - (1+\mu)[(2+\mu) - x_1 x_2] + (1+\mu) = 1.$$

∴

$$x_1^2 x_2^2 + (1+\mu)x_1 x_2 - (\mu + 2) = 0, \quad (2.4)$$

so that

$$x_1 x_2 = -\frac{1}{2}(1+\mu) \pm \frac{1}{2}[(1+\mu)^2 + 4(\mu+2)]^{1/2}$$

Hence

$$x_1 x_2 = -\frac{1}{2}(1+\mu) \pm \frac{1}{2}(\mu+3)$$
$$= -\frac{1}{2}\Big(1 + \mu \mp (\mu+3)\Big),$$

$$(x_1x_2)_+ = 1 \quad (\text{p.d. off } \pm\sqrt{\mu}), \tag{2.5a}$$
$$(x_1x_2)_- = -(\mu+2) \quad (\text{p.d. off } x=0). \tag{2.5b}$$

Substituting (10.5) into (10.4) as $(x_1+x_2)^2 - x_1x_2 = 2+\mu$ gives

$$(a) \quad (x_1x_2)_+ \quad (x_1+x_2)^2 = 3+\mu$$

$$(b) \quad (x_1x_2)_- \quad (x_1+x_2)^2 = 0 \quad \Rightarrow x_1 = -x_2.$$

(a) : $x_1+x_2 = \pm\sqrt{3+\mu}$, $x_1x_2 = 1$. Choosing the $+$ sign, by the property of quadratics we have that

$$x^2 - \sqrt{3+\mu}x + 1 = 0,$$

so that

$$x_{1,2} = \frac{1}{2}\sqrt{3+\mu} \pm \frac{1}{2}[\mu-1]^{1/2}.$$

Therefore, when $\mu = 1$, $x_1 = x_2 = \frac{1}{2}\sqrt{3+1} = 1$ and we have a P2-cycle (associated with $\pm\sqrt{\mu}$).

(b) : If $x_1+x_2 = 0$, then $x_1x_2 = (\mu+2) \Rightarrow x_1^2 = 2+\mu$. Therefore

$$x_1 = \pm\sqrt{2+\mu} = \mp x_2.$$

At $\mu = -2$, $x_1 = x_2 = 0$ ie. P2-cycle (associated with $x = 0$). Note that $2+\mu \geq 0$ for real values of x_1 and x_2.

Stability of P2 cycles. From the previous lecture we have that

$$\lambda_{P2} = f'(x_1)f'(x_2)$$

where $f' = 1+\mu-3x^2$.
Therefore for (a):

$$\lambda_{p_2} = (1+\mu-3x_1^2)(1+\mu-3x_2^2)$$
$$= (1+\mu)^2 - 3(1+\mu)(x_1^2+x_2^2) + 9x_1^2x_2^2$$
$$= (1+\mu)^2 - 3(1+\mu)^2 + 9$$
$$= 9 - 2(1+\mu)^2.$$

We therefore have stability for

$$-1 < 9 - 2(1+\mu)^2 < 1$$

$$5 > (1+\mu)^2 > 4$$

$$\text{i.e.} \quad \sqrt{5} - 1 > \mu > 1 \quad \text{since} \quad (1 + \mu > 0).$$

$\mu = \sqrt{5} - 1$ corresponds to the onset of a P4-cycle, while $\mu = 1$ corresponds to a fold bifurcation for P2.

For (b) $x_1 = -x_2 = \sqrt{2+\mu}$

$$\lambda_{P2} = (1+\mu)^2 - 6(1+\mu)(2+\mu) + 9(2+\mu)^2$$

$$= [(1+\mu) - 3(2+\mu)]^2$$

$$= (5+2\mu)^2.$$

Since $2 + \mu \geq 0$, $5 + 4\mu \geq 1$ and so this P2-cycle is unstable, and coexists with the stable $x = 0$ state. This is a subcritical period-doubling bifurcation.