LECTURE 12: HENON MAP

The 2-D Henon map is defined as

$$x_{n+1} = 1 - ax_n^2 + y_n = f(x_n, y_n), \quad (4.1a)$$

$$y_{n+1} = bx_n \quad = g(x_n, y_n). \quad (4.1b)$$

Fixed points are given by:

$$x = 1 - ax^2 + y, \quad y = bx,$$

so that

$$x = 1 - ax^2 + bx.$$

Therefore

$$ax^2 + (1-b)x - 1 = 0,$$

which gives

$$x_\pm = -\frac{1-b}{2a} \pm \frac{1}{2a}[(1-b)^2 + 4a]^{1/2}, \quad (4.2)$$

provided $(1-b)^2 + 4a > 0$.

Hence there are two fixed points for $a \geq -1/4(1-b)^2$ and no fixed points for $a < -1/4(1-b)^2$.

We therefore have a saddle-node bifurcation when $a < 0$ and $x, y > 0$ at

$$a = a_0 = -\frac{1}{4}(1-b)^2, \quad (4.3a)$$

$$x = x_0 = -\frac{(1-b)}{2a_0} = \frac{2}{1-b}, \quad (4.3b)$$

$$y = y_0 = \frac{2b}{1-b} \quad (b \neq 1). \quad (4.3c)$$

Stability

We compute the 2×2 Jacobian matrix

$$J(\underline{X}) = \left[\frac{\partial f_i}{\partial x_j}\right]_{\underline{x}_\pm} = \begin{bmatrix} -2ax_\pm & 1 \\ b & 0 \end{bmatrix},$$

where $f_1 = f$, $f_2 = g$.
The characteristic equation $|J - \lambda I_2| = 0$ gives

$$\lambda^2 + 2ax_{\pm}\lambda - b = 0 \Rightarrow \quad \lambda_{1,2} = -ax_{\pm} \pm [a^2x_{\pm}^2 + b]^{1/2}. \tag{4.4}$$

NB: Since $\det J = -b$ for all x, the map is a uniform contraction for $|b| < 1$, a uniform expansion for $|b| > 1$ and area-preserving for $b = \pm 1$ (uniform because b is fixed).
We shall focus on $|b| < 1 (\Rightarrow x_0, y_0 > 0$, and $a_0 < 0$ at the saddle-node bifurcation.
Substituting for a_0 into (12.2):

$$x_{\pm} = -\frac{1}{2a}(1-b) \pm \frac{1}{a}[a - a_0]^{1/2}. \tag{4.5}$$

It can be shown that x_+ is stable for $a_0 < a < a_1$, where $a_1 = \frac{3}{4}(1-b)^2$ and $|b| < 1$.
When $a = a_1$, x_+ undergoes a period-doubling bifurcation. (When $a = a_1$, $x_+ = \frac{2}{3(1-b)}$.)
x_- is always unstable.

4.1 Stability Analysis of Fixed Points of Henon Map

$\underline{x_+}$

x_+ is stable for $-1 < \lambda_{1,2} < 1$. When

$$\lambda_{1,2} = -1 \quad \Rightarrow \text{p.d. bifurcation}$$

$$\lambda_{1,2} = +1 \quad \Rightarrow \text{fold bifurcation.}$$

Suppose $\lambda_j = -1$, $(j = 1, 2)$. Then

$$-1 = -ax_+ + [a^2x_+^2 + b]^{1/2}$$

so that

$$ax_+ - 1 = (a^2x_x^2 + b)^{1/2} > 0,$$

since $ax_+ - 1 > 0$.
Therefore squaring gives

$$a^2x_+^2 - 2ax_+ + 1 = a^2x_+^2 + b$$

so that

$$x_+ = \frac{1-b}{2a},$$

and we have a period-doubling bifurcation. (12.4) then becomes

$$\frac{1-b}{2a} = -\frac{1-b}{2a} + \frac{1}{a}(a-a_0)^{1/2},$$

which simplifies to give

$$a = a_1 = \frac{3}{4}(1-b)^2 > a_0. \tag{4.6}$$

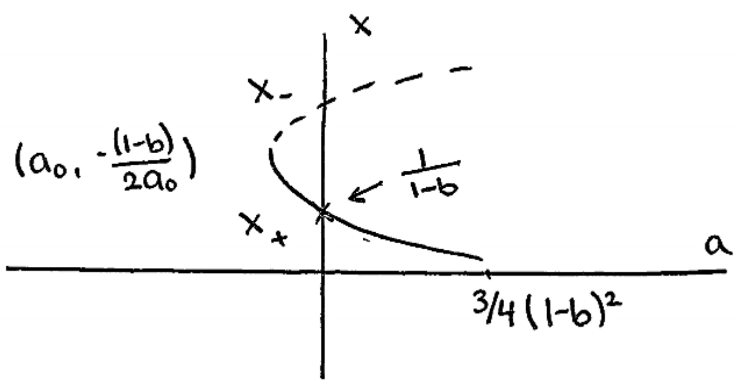
When $\lambda_j = 1$, $j = 1, 2$, we obtain the original saddle-node fold bifurcation at $a = a_0$:

$$x_+ = -\frac{1-b}{2a}.$$

Therefore x_+ is stable for $a_0 < a < a_1$.
N.B. Since we have a saddle-node bifurcation, x_- is always unstable.

4.2 2-Cycles of the Henon Map

The picture so far:



Since $a_0 < 0$, and $b < 1$, (12.5) shows that, in the neighbourhood of a_0, x_+ forms the lower locus of fixed points and x_- the upper locus, as $|x_+| < |x_-|$.
We can show that the period-two cycles satisfy

$$X_{1,2} = \frac{1}{2a}(1-b) \pm \frac{1}{2a}[4a - 3(1-b)^2]^{1/2}, \tag{4.7}$$

provided $a > \frac{3}{4}(1-b)^2$.

When $a = a_1 = \frac{3}{4}(1-b)^2 (\geq 0)$, $X_1 = X_2 = \frac{2}{3}(1-b) > 0$, since $|b| < 1$.
The stability of the 2-cycle can be determined from

$$K(X) = J(X_1)J(X_2) = \begin{bmatrix} -2aX_1 & 1 \\ b & 0 \end{bmatrix} \begin{bmatrix} -2aX_2 & 1 \\ b & 0 \end{bmatrix}.$$

We can show that the 2-cycle is stable in $a_1 < a < a_2$, where

$$a_2 = (1-b)^2 + \frac{1}{4}(a+b)^2, \tag{4.8}$$

for $|b| < 1$.

The Inverse Map The Henon map is invertible with inverse

$$y_n = x_{n+1} + \frac{a}{b^2}y_{n+1}^2 - 1 \tag{4.9a}$$

$$x_n = \frac{1}{n}y_{n+1}, \tag{4.9b}$$

with $\det(\text{Jacobian}) = -b^{-1}$.

4.3 Standard (or Chirikov) Map

Consider

$$I_{n+1} = I_n + K\sin\theta_n = f(\theta_n, I_n) \tag{4.10a}$$

$$\theta_{n+1} = \theta_n + I_{n+1} = I_n + \theta_n + K\sin\theta_n = g(\theta_n, I_n), \tag{4.10b}$$

where I_n and θ_n are calculated mod2π and $K > 0$.

(i) The map is area-preserving:

$$J = \begin{bmatrix} f_{I_n} & f_{\theta_n} \\ g_{I_n} & g_{\theta_n} \end{bmatrix} = \begin{bmatrix} 1 & K\cos\theta_n \\ 1 & 1+K\cos\theta_n \end{bmatrix}.$$

Therefore $\det J = 1$, from which follows area-preserving.

(ii) Fixed points and Stability:

$I = I + K\sin\theta \quad \Rightarrow \quad \sin\theta = 0 \quad \Rightarrow \quad \theta = r\pi$.

$\theta = I + \theta + K\sin\theta \quad \Rightarrow \quad I = 0$.

Therefore the fixed points are $(I, \theta) = (0, \pi)$ and $(0, 0)$, since the solutions are conjugate mod2π.

At $(0, 0)$:

$$J_0 = \begin{bmatrix} 1 & K \\ 1 & 1+K \end{bmatrix},$$

$$\Rightarrow \quad \lambda^2 - (2+K)\lambda + 1 = 0,$$

$$\Rightarrow \quad \lambda_0 = \frac{1}{2}(2+K) \pm \frac{1}{2}\ sqrt(2+K)^2 - 4,$$

$$\Rightarrow \quad \lambda_0 = \frac{1}{2}\left\{(2+K) \pm \sqrt{K(K+4)}\right\}.$$

Since $K \geq 0$, λ_0 is always real and $(0,0)$ is stable for

$$-1 < \lambda_0^{\pm} < 1,$$

i.e. for

$$-(4+K) < \pm\sqrt{K(K+4)} < -K.$$

Since $K \geq 0$, $-(4+K) < 0$ and $\sqrt{K(K+4)} \geq 0$, the $+\sqrt{}$ contradicts this inequality, and hence $(0,0)$ is unstable.

At $(0,\pi)$:

$$J_\pi = \begin{bmatrix} 1 & -K \\ 1 & 1-K \end{bmatrix},$$

so that

$$\Lambda^2 - (2-K)\Lambda + 1 = 0.$$

Hence

$$\Lambda = \frac{1}{2}(2-K) \pm \frac{1}{2}[(2-K)^2 - 4]$$
$$= \frac{1}{2}\left\{(2-K) \pm [K(K-4)]^{1/2}\right\},$$

which is stable for $-1 < \Lambda_\pi^{\pm} < 1$.

For $0 \leq K < 4$, $\Lambda_\pi^{\pm} \in \mathcal{C}$ with real part $\frac{2-K}{2}$.

This will be stable for

$$-1 < \frac{1}{2}(2-K) < 1,$$

or

$$4 > K > 0.$$