LECTURE 13:

INTRODUCTION TO CHAOS: THE LOGISTIC MAP

In 1976, Robert May proposed the following 1-D map as a model for population dynamics:

$$x_{n+1} = \mu x_n(1 - x_n) = f(x_n, \mu), \tag{1.1}$$

for $0 \leq x_n \leq 1$ and $0 \leq \mu \leq 4$. This is known as the logistic map. We consider the bifurcations of (1.1) as μ increases from 0.

(a)The 1st bifurcation is a transcritical bifurcation

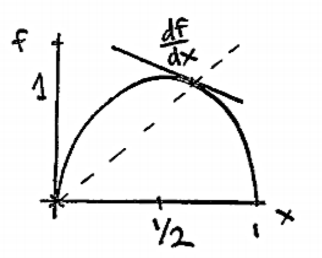
Consider the fixed points of (1.1)

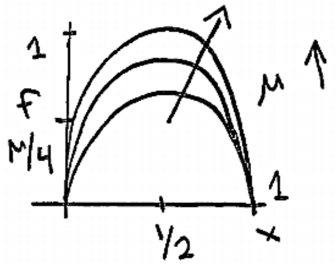
$$x = f(x, \mu) = \mu x(1 - x) \quad \Rightarrow \quad x_0 = 0 \quad \text{or} \quad x_e = 1 - \frac{1}{\mu}, \tag{1.2}$$

for $\mu > 0$.

Multiplier:

$$\lambda = \left.\frac{df}{dx}\right|_{x_e} = \mu - 2\mu x_e.$$





For $x_0 = 0$, $\lambda_0 = \mu$; and for $x_e = 1 - \frac{1}{\mu}$, $\lambda_e = 2 - \mu$.

Stability occurs for $|\lambda| < 1$, and we are interested in $\mu \geq 0$.

At $\mu = 0$, $x_0 = 0$ is the only fixed point. For $\mu > 0$ there are two fixed points, given by (13.2).

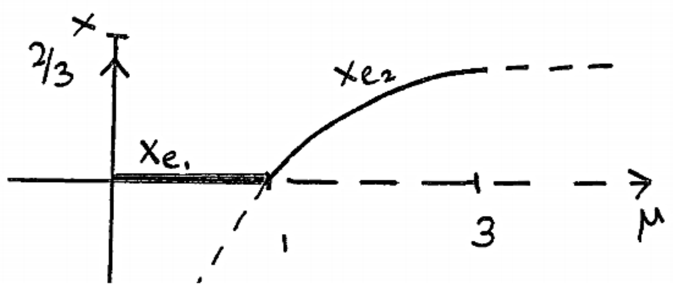
When $\mu = 1$, $\lambda_0 = \lambda_e = 1$ and $x_e = 1 - \frac{1}{\mu} = 0$.

$\therefore$ x_0 loses stability to the branch $x_e = 1 - \frac{1}{\mu}$ in a transcritical bifurcation.

As μ increases from 1 to 3, λ decreases from $+1$ to -1 and x_e increases from 0 to $2/3$.

$x_0 = 0$ is stable for $0 \leq \mu(= \lambda_0) < 1$ and unstable for $\mu > 1$.

x_e is stable for $-1 < 2 - \mu(= \lambda_e) < 1$, ie. for $1 < \mu < 3$.



(b) The 2nd bifurcation from x_e is a p.d. bifurcation

When $\mu = 3$, $\lambda_e = -1$ and $x_e = \frac{2}{3}$ loses stability to a period-2 cycle via a p.d. bifurcation.

Let the points on the period-2 cycle be x_1, x_2, i.e.

$$x_2 = \mu x_1(1 - x_1) \quad \text{and} \quad x_1 = \mu x_2(1 - x_2). \tag{1.3}$$

They are fixed points of $x_j = f^2(x_j, \mu)$ for $j = 1, 2$.

To find x_1 and x_2:

From (13.3)

$$x_2 - x_1 = \mu(x_1 - x_2) + \mu(x_2^2 - x_1^2),$$

and since $x_1 \neq x_2$ it follows that

$$x_1 + x_2 = \frac{1+\mu}{\mu}. \tag{1.4}$$

Also (13.3) gives

$$x_1 x_2 = \mu^2 x_1 x_2 (1 - x_1)(1 - x_2)$$

so that

$$x_1 x_2 = \frac{\mu + 1}{\mu^2}. \tag{1.5}$$

Therefore x_1 and x_2 satisfy

$$\mu^2 x^2 - \mu(1+\mu)x + (1+\mu) = 0,$$

with solution

$$x_{1,2} = \frac{(1+\mu)}{2\mu} \pm \frac{1}{2\mu}[(1+\mu)(\mu - 3)]^{1/2}, \tag{1.6}$$

which is real for $\mu > 3$.

The multiplier $\lambda_{P2} = f'(x_1)f'(x_2)$, where $f'(x) = \mu - 2\mu x$.

Therefore

$$\lambda_{P2} = -\mu^2 + 2\mu + 4.$$

When $\mu = 3$ (on set of the P2 cycle), $\lambda_{P2} = 1$, and the P2 cycle comes in as a fold bifurcation, with stability maintained until $\lambda = -1$, ie. $\mu = 1 + \sqrt{6}$ (we take the positive square root since $\mu > 0$).

1.5 THE FAMILY OF LOGISTIC MAPS

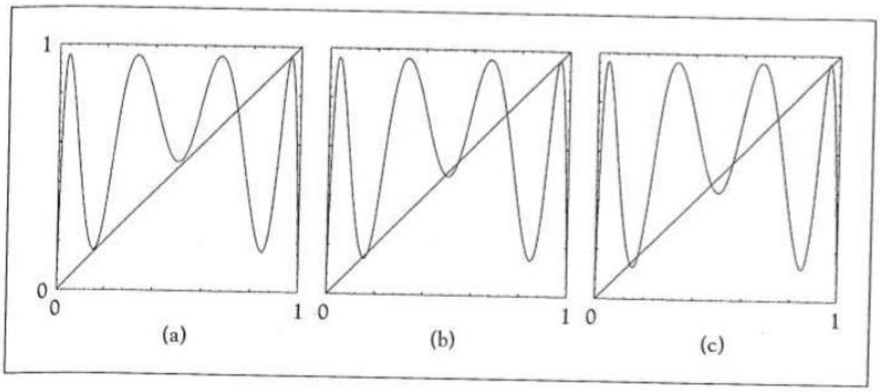
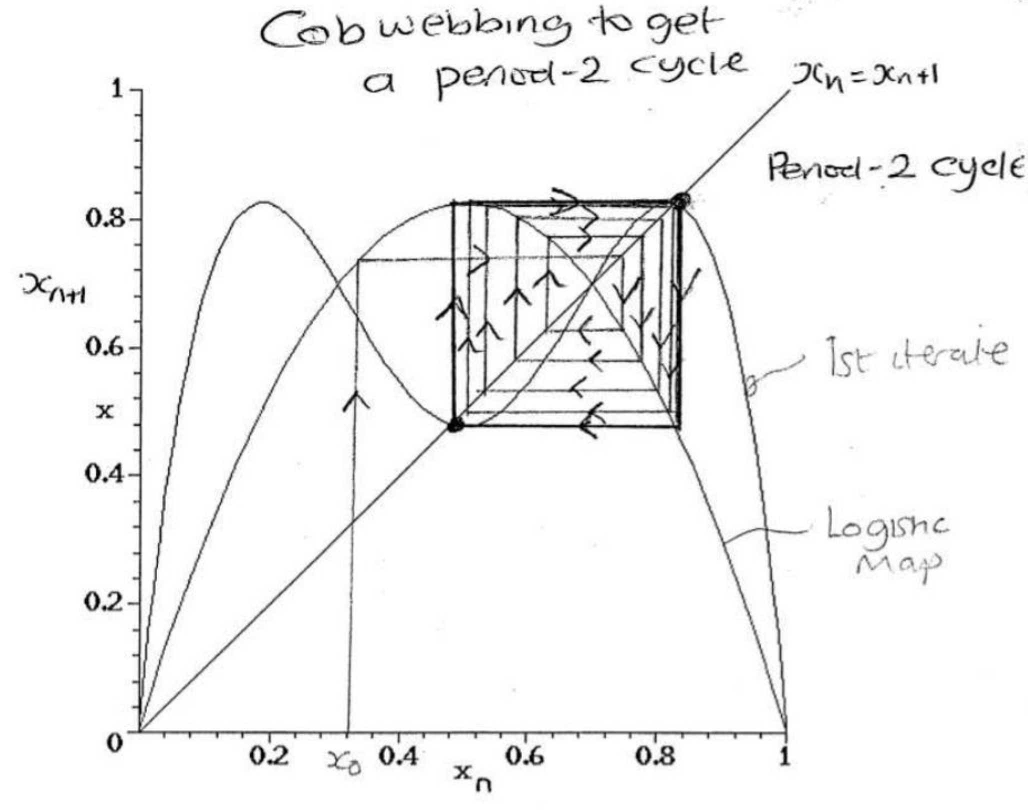
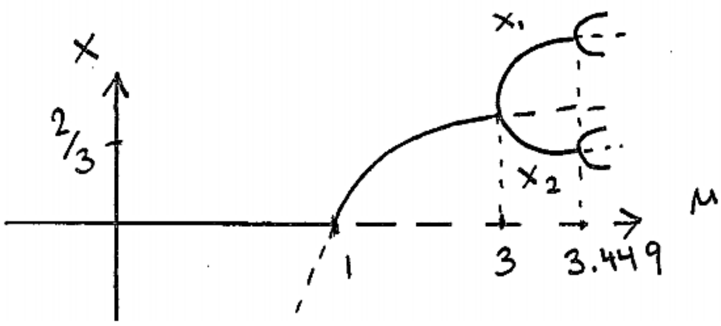


Figure 1.9 Graphs of the third iteration $g^3(x)$ of the logistic map $g_a(x) = ax(1-x)$.
Three different parameter values are shown: (a) $a = 3.82$ (b) $a = 3.84$ (c) $a = 3.86$.



The P2 cycle loses stability at $\mu = 1 + \sqrt{6} \approx 3.44949$ via a p.d. bifurcation to a cycle of period 4.



NB. (a) Both x_1 and x_2 undergo p.d. bifurcations at $\mu = 3$.
To find the fixed points of the P4 cycle we must solve $f^4(x, \mu) = x$.
(b) The p4 cycle first appears at $\mu = 3.449$ when its multiplier

$$\lambda_{P4} = f'(x_1) \dots f'(x_4) = 1$$

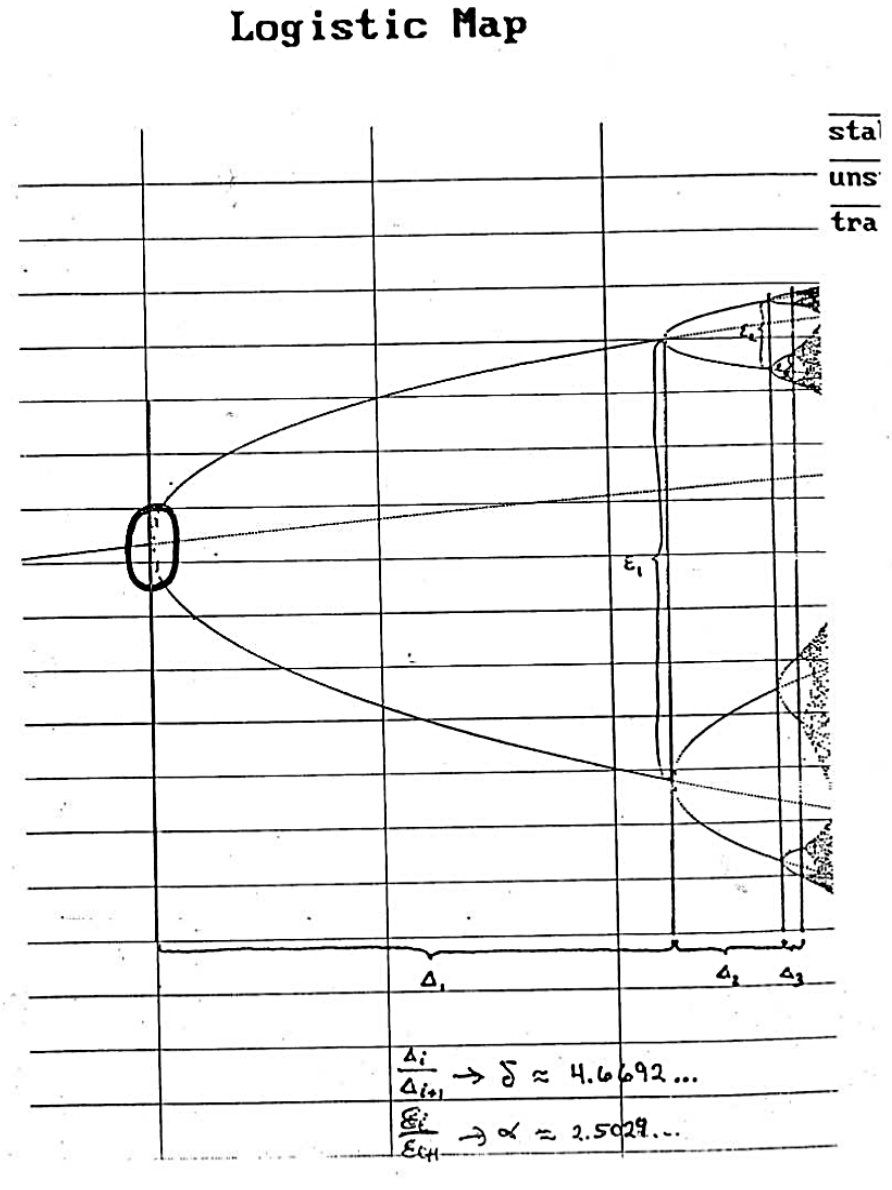
and loses stability when $\lambda_{P4} = -1$ to a P8 cycle when $\mu = 3.54409$.

appearing to converge to a limit μ_∞ as a geometric progression like

$$\mu_{p_{2^k}} = \mu_\infty - c\tau^{-k} \tag{1.7}$$

where

$$\mu_\infty = 3.569946\ldots, \quad c = 2.6327\ldots, \quad \tau = 4.669202\ldots \quad . \tag{1.8}$$



For $\mu > \mu_\infty$, chaotic behaviour occurs.

This pattern of p.d. cascades and values in (13.8) is universal (see Feigenbaum's work for a large class of unimodal maps. The interval $(\mu_\infty, 4)$ contains an ∞ number of sma windows of μ-values for which there exists stable m-cycles. The first cycles appear fc $\mu > \mu_\infty$ and are of even period. Next odd cycles appear in descending order (accordin Sarkovskii's Theorem) until a period-3 cycle appears.

Theorem 1 (Sarkovskii's Theorem). *Consider the following ordering of all positiv integers:*

$1 \triangleleft 2 \triangleleft 2^2 \triangleleft \ldots \triangleleft 2^k \triangleleft 2^{k+1} \triangleleft \ldots \triangleleft 2^k(2l+1) \triangleleft 2^k(2l-1) \triangleleft \ldots \triangleleft 2^k 5 \triangleleft 2^k 3 \triangleleft \ldots \triangleleft 2(2l+1)$
$2(2l-1) \triangleleft \ldots \triangleleft 2.5 \triangleleft 2.3 \triangleleft (2l+1) \triangleleft (2l-1) \triangleleft \ldots \triangleleft 5 \triangleleft 3 \ldots.$

Suppose f is a continuous map of an interval to itself with a periodic orbit of period with $q \triangleleft p$ is the above ordering. Then f already has a periodic orbit of period q.

Examples

1. If f has a P8 orbit, it already has P2 and P4 orbits.
2. If f has a P3 cycle, then it already has orbits of arbitrary long period ie. it ha chaotic transients.

This occurs for $\mu = 3.841499$ (see 'Period 3 implies chaos' by Li and Yorke).
When $\mu = 4$, periods of all orders are present and they are all unstable. The logisti map is then

$$x_{n+1} = 4x_n(1 - x_n).$$